Combinatorial certificates in SDP duality: how elementary row operations help

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Parts joint work with Minghui Liu, Alex Touzov, Melody Zhu, Quoc Tran-Dinh

A Semidefinite Program (SDP)

$$\begin{aligned} \inf_X \ & C \bullet X \\ s.t. \ & X \succeq 0 \\ & A_i \bullet X = b_i \, (i = 1, \ldots, m). \end{aligned}$$

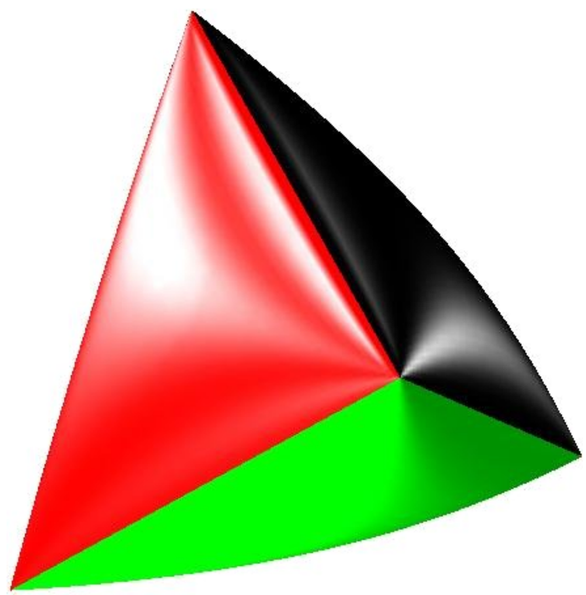
Here

- A_i, C are symmetric data matrices, X is variable matrix.
- For symmetric matrices $S, T : S \preceq T$ means: $T - S$ is positive semidefinite (psd).
- $S \bullet T = \sum_{i,j} s_{ij} t_{ij}$.
- $X \succeq 0 \overset{\text{def}}{\Leftrightarrow}$ all principal subdeterminants are nonnegative.
- Equivalently, if $v^T X v \geq 0 \, \forall v \in \mathbb{R}^n$.

Example: 3×3 correlation matrices: the elliptope

The set $\{X \succeq 0,\ \mathrm{Diag}(X) = I_3\}$

can be represented as $\left\{ (u, v, w) \;\middle|\; \begin{pmatrix} 1 & u & v \\ u & 1 & w \\ v & w & 1 \end{pmatrix} \succeq 0 \right\}$



Why is SDP important:
LP $\subseteq$ SDP $\subseteq$ Convex Optimization

LP as SDP:

- **If A_i and C are diagonal $\Rightarrow$ we can assume so is X.**
- **And, diagonal matrix is psd $\Leftrightarrow$ diagonal elements are non-negative.**
- **So LP can be modeled as SDP: make A_i, C diagonal.**
- **(Or: explicitly force offdiagonal elements of X to $\mathbf{0}$).**

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- So LP can be modeled as SDP: make A_i, C diagonal.
- (Or: explicitly force offdiagonal elements of X to $\mathbf{0}$).

SDP is a convex problem:

- Feasible set is convex, since set of psd matrices is.

Why is SDP important: applications in

- 0–1 Integer programming.
- Approx algorithms
- Chemical engineering
- Chemistry
- Coding theory
- Control theory
- Combinatorial opt
- Discrete geometry
- Eigenvalue optimization
- Facility planning
- Finance
- Geometric optimization
- Global optimization
- Graph visualization
- Inventory theory
- Machine learning
- Matrix analysis
- PDEs
- Probability theory
- Robust optimization
- Signal processing
- Statistics
- Structural optimization
- Several thousand (up to 20 thousand!) papers on SDP in the last 30 years.

Other "application"

- Gives good insight into convex optimization.
- Basics are just like in LP.
- But: What can go wrong in convex optimization already can go wrong in SDP.

Other "application"

- **Gives good insight into convex optimization.**
- **Basics are just like in LP.**
- **But: What can go wrong in convex optimization already can go wrong in SDP.**
- **Precisely: to solve an SDP, we need:**
  - **A dual problem**
  - **An infeasibility certificate**
- **These are not quite the way we would like them to be.**

SDP duality

$$\begin{array}{llll} & \inf_X \ C \bullet X & \qquad\qquad & \sup_y \ b^T y \\ (P) & s.t. \ X \succeq 0 & & \sum_{i=1}^m y_i A_i \preceq C \ \ (D) \\ & A_i \bullet X = b_i \, (i \in [m]). & & \end{array}$$

Easy: If X and y are feasible, then $C \bullet X \geq b^T y$.

But:

- unattained optimal values
- positive duality gaps
- Slater's condition helps, but it may not hold.
- SDP is poly time solvable only under restrictive assumptions

Alternative system (Farkas' lemma) to prove infeasibility

$$
(P) \quad \begin{array}{ll} \inf_X & C \bullet X \\ s.t. & X \succeq 0 \\ & A_i \bullet X = b_i \, (i \in [m]). \end{array}
\qquad\qquad
\begin{array}{l} \sum_{i=1}^m y_i A_i \succeq 0 \\ b^\top y = -1 \end{array} \quad (ALT\text{-}P)
$$

Easy: (P) infeasible $\Leftarrow$ $(ALT\text{-}P)$ feasible

But:

- $\not\Rightarrow$
- Thus $(ALT\text{-}P)$ is not useful for complexity
- not known whether feasibility of (P) can be decided in polynomial time

Some partial history of duality in conic linear programs:

- **Duffin, 1956:** **Infinite programs**
- **Berman, 1973:** **Cones, matrices, and mathematical programming**
- **Ben-Israel, Charnes, Kortanek, 1968:** **Duality and asymptotic solvability over cones**
- **Berman, Ben-Israel, 1971:** **Duality and asymptotic solvability over cones**
- $\vdots$

Better understanding of in the last 15+ years based on:

(1) On the closedness of the linear image of a closed convex cone, P, 2007

(2) Facial reduction: Borwein-Wolkowicz 80-81; Waki-Muramatsu 2012; P 2000, 2013, Liu-P 2016

(3) Elementary row operations

Elementary row operations for linear systems:

$Ax = b$ is infeasible $\Leftrightarrow$ by elementary row operations it can be transformed into

$$\begin{aligned} A'x &= b' \\ 0^T x &= 1 \end{aligned}$$

- $\Leftarrow$ is easy, so a good certificate
- Can be used to generate any infeasible system

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We will use these operations to understand SDPs:

Part 1 Badly behaved feasible SDPs

Part 2 Infeasible SDPs

Part 3 Asymptotic feasibility, weak infeasibility

Part 4 Ramana's perfect infeasibility certificate

Reformulations

$$\begin{array}{llll} & \inf_X \ C \bullet X & \qquad\qquad & \sup_y \ b^T y \\ (P) & s.t. \ X \succeq 0 & & \sum_{i=1}^m y_i A_i \preceq C \ \ (D) \\ & A_i \bullet X = b_i \, (i \in [m]). & & \end{array}$$

We obtain a reformulation of (P) − (D) by a sequence of the following:

- Elementary row operations on the equations of (P)
- For $\lambda \in \mathbb{R}^m$

$$C \leftarrow C + \sum_i \lambda_i A_i$$

- For a V invertible matrix

$$A_i \leftarrow V^T A_i V \ (i \in [m]), C \leftarrow V^T C V,$$

Part 1: Why all bad SDPs look the same

Dual :

$$\begin{array}{ll} \sup & 2y_1 \\ s.t. & y_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \sup & 2y_1 \\ s.t. & \begin{pmatrix} 1 & -y_1 \\ -y_1 & 0 \end{pmatrix} \succeq 0 \end{array}$$

Only feasible y_1 **is** $y_1 = 0.$

Primal (with variable matrix X**) is equivalent to:**

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Only feasible y_1 is $y_1 = 0$.

Primal (with variable matrix X) is equivalent to:

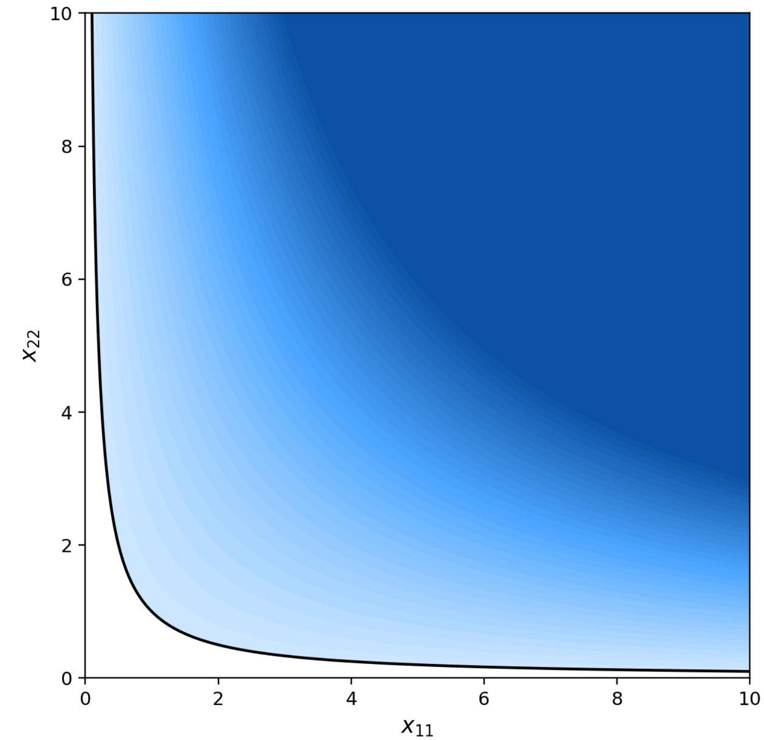
$$\begin{array}{ll} \inf & x_{11} \\ s.t. & x_{11} \geq 0,\ x_{22} \geq 0,\ x_{11}x_{22} \geq 1. \end{array}$$

Part 1: Why all bad SDPs look the same

But

$$\inf\ x_{11}$$
$$s.t.\ x_{11} \geq 0,\ x_{22} \geq 0,\ x_{11}x_{22} \geq 1.$$

has unattained infimum!



Part 1: Why all bad SDPs look the same

- **Semidefinite system:**

$$(D_{sys}) \quad \sum_{i=1}^{m} y_i A_i \preceq C$$

We say (D_{sys}) is badly behaved, if $\exists b$ s.t.

$$\sup\{\, b^\top y \mid y \in D_{sys} \,\}$$

is finite, but (P) has no solution with the same value.

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- **A slack matrix in (D) is a matrix**

$$Z := C - \sum_{i=1}^m y_i A_i \succeq 0.$$

Part 1: Why all bad SDPs look the same

One variable case: In

$$(D_{sys}) \quad y_1 \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & \underbrace{V_{22}}_{n-r} \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

suppose rhs is the maximum rank slack. Then (D_{sys}) is badly behaved $\Leftrightarrow$

$$V_{22} \succeq 0, \; R(V_{12}^T) \not\subseteq R(V_{22}).$$

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So, just a larger version of

$$y_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Part 1: Why all bad SDPs look the same

Theorem: Assume in

$$(D_{sys}) \quad \sum_{i=1}^{m} y_i A_i \preceq C$$

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Then (D_{sys}) is badly behaved $\Leftrightarrow \exists$ reformulation

$$(D'_{sys}) \quad \sum_{i=1}^{m} y_i A'_i \preceq C$$

s.t.

$$y_1 A'_1 \preceq C$$

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is badly behaved.

I.e., one variable bad subsystem certifies bad behavior.

Part 1: Why all bad SDPs look the same

Example

$$y_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has positive gap, when we seek $\sup y_2$.

Part 1: Why all bad SDPs look the same

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One variable bad subsystem:

$$\cancel{y_1} \cancel{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} + y_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Part 2: Infeasibility

Example

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} \bullet X = \underbrace{0}_{b_1}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A_2} \bullet X = \underbrace{-1}_{b_2}$$

$$X \succeq 0$$

Farkas lemma certificate fails: there is no $y \in \mathbb{R}^2$ s.t.

$$y_1 A_1 + y_2 A_2 \succeq 0, \; y_1 b_1 + y_2 b_2 = -1.$$

Part 2: Infeasibility

Example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$

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$$X \succeq 0$$

Still, it is easy to see why it is infeasible:

- Suppose X feasible $\Rightarrow x_{11} = 0$
  $\Rightarrow x_{12} = x_{13} = 0$
  $\Rightarrow x_{22} = -1$, contradiction!

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We will find such a structure in every infeasible semidefinite system.

Some linear algebra

$$\begin{pmatrix} \boldsymbol{I_r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bullet \boldsymbol{X} = \mathbf{0},\ \boldsymbol{X} \succeq \mathbf{0} \Rightarrow ?$$

Some linear algebra

$$\begin{pmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \bullet \boldsymbol{X} = \boldsymbol{0},\ \boldsymbol{X} \succeq \boldsymbol{0} \Rightarrow \boldsymbol{X} = \begin{pmatrix} \boldsymbol{0}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_{22} \end{pmatrix},\ \boldsymbol{X}_{22} \succeq \boldsymbol{0}.$$

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{array}{rcl} A'_i \bullet X & = & 0 \ (i = 1, \ldots, k) \\ A'_{k+1} \bullet X & = & -1 \\ & \vdots & \\ X & \succeq & 0 \end{array} \qquad (\mathrm{P}_{\mathrm{ref,infeas}})$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$

$$A'_1 = \begin{array}{c} \begin{array}{cc} \overbrace{}^{r_1} & \overbrace{}^{n-r_1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad A'_i = \begin{array}{c} \begin{array}{ccc} \overbrace{}^{r_1+\ldots+r_{i-1}} & \overbrace{}^{r_i} & \overbrace{}^{n-r_1-\ldots-r_i} \end{array} \\ \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \end{array}$$

with $r_i \geq 0 \ \forall i$.

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$$

with $r_i \geq 0 \ \forall i$. Liu-P, 2015

Proof of " $\Leftarrow$ " : Suppose X feasible in $(\mathrm{P}_{\mathrm{ref,infeas}})$

$\Rightarrow$ first r_1 rows of X are 0

$\dots$

$\Rightarrow$ first $r_1 + \ldots + r_k$ rows of X are 0

$\Rightarrow$ trace of diag. block of X is -1, contradiction!

Proof of " $\Rightarrow$ " : Also not hard P-Touzov, 2022

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

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\begin{aligned}
A_i' \bullet X &= 0 \; (i = 1, \dots, k) \\
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where $k \geq 0$, and for $i = 1, \dots, k+1$

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$$

with $r_i \geq 0 \; \forall i$. Liu-P, 2015

A normal form of empty spectrahedra

Application 1: generating infeasible SDPs

By this theorem, we can generate any infeasible SDP, as:

(1) Generate a system like $(\mathrm{P}_{\mathrm{ref}})$

(2) Reformulate it.

There are only finitely many representative infeasible SDPs!

How many? # of partitions of $\{1, \ldots, n\}$ into nonempty subsets

Problem library by Liu-P 2016: infeasible SDPs

Application 2: better understanding of polynomial optimization

Example: Motzkin polynomial

$$f(x, y) = 1 - 3x^2y^2 + x^2y^4 + x^4y^2$$

It is ≥ 0

To certify nonnegativity, we can try to find a sum of squares (SOS) decomposition $f = \sum_i (f_i)^2$ for some f_i polynomials.

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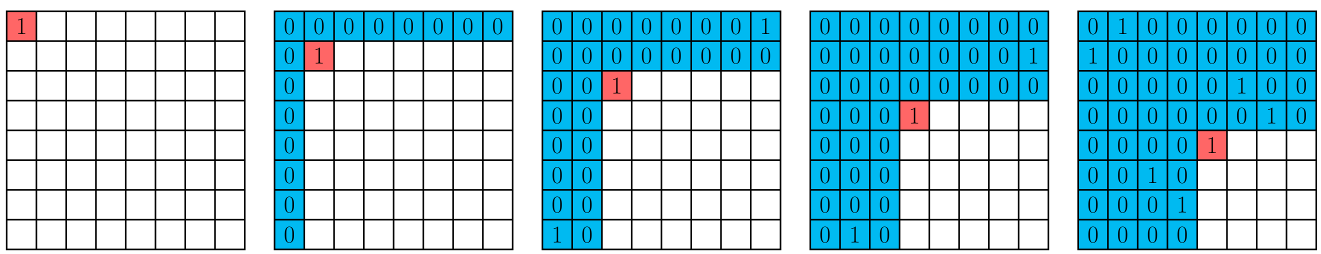
Checking SOSness is an SDP (Lasserre, Parrilo)

f is not SOS $\Rightarrow$ The SDP is infeasible, and in the form of $(\mathrm{P}_{\mathrm{ref,infeas}})$ without any reformulation



A_1 A_2 A_3 A_4 A_5

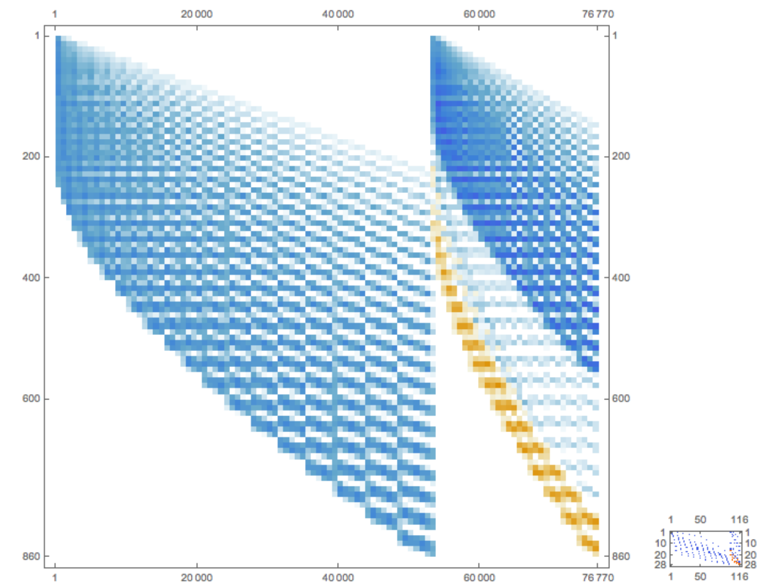
with $(b_1, b_2, b_3, b_4, b_5) = (0, 0, 0, 0, -3)$

Paper: P-Touzov, 2022, FOCM

Application 3: recognizing infeasibility in practice

- Sometimes we do not even have to reformulate an SDP to find the trivial structure that proves infeasibilty ...or to reduce the SDP.

- Zhu–P–Tran-Dinh, 2018 Fast Sieve-SDP preprocessor
- Before and after picture of an SDP



Though this is a best case example, many polynomial optimization problems get greatly reduced.

Part 3: Asymptotic feasibility, weak infeasibility

- **Recall dual, without objective function:**

$$(D) \quad \sum_{i=1}^{m} y_i A_i \preceq C$$

- **We say (D) is asymptotically feasible, if an arbitrarily small perturbation of C makes it feasible.**

- **Feasible $\Rightarrow$ asymptotically feasible. But: $\not\Leftarrow$**

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- **We say (D) is asymptotically feasible, if an arbitrarily small perturbation of C makes it feasible.**

- **Feasible $\Rightarrow$ asymptotically feasible. But: $\not\Leftarrow$**

- **We say (D) is weakly infeasible, if infeasible, but asymptotically feasible.**

- **Weakly infeasible SDPs are hard: often mistaken for feasible ones by solvers**

Part 3: Asymptotic feasibility, weak infeasibility

Example:

$$y_1 \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} + y_2 \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A_2} \preceq \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{C}$$

is weakly infeasible.

Infeasible:

$$S := C - y_1 A_1 - y_2 A_2 = \begin{pmatrix} -y_1 & 0 & -y_2 \\ 0 & -y_2 & 1 \\ -y_2 & 1 & 0 \end{pmatrix} \not\succeq 0$$

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Asymptotically feasible: for any $\epsilon > 0$

$$S_\epsilon := C - y_1 A_1 - y_2 A_2 = \begin{pmatrix} -y_1 & 0 & -y_2 \\ 0 & -y_2 & 1 \\ -y_2 & 1 & \epsilon \end{pmatrix} \succ 0 :$$

choose y_2 to make lower right 2×2 block $\succ 0$; then choose y_1 to make $S_\epsilon \succ 0$.

Theorem: (D) asymptotically feasible $\Leftrightarrow$ it has a reformulation

$$\sum_{i=1}^m y_i A_i' \preceq C'$$

where $0 \le \ell \le m$ and for $i = 1, \dots, \ell$

$$A_1' = \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix}, \quad A_i' = \begin{pmatrix} \overbrace{\times}^{r_1+\dots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\dots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix},$$

and

$$C' = \begin{pmatrix} \overbrace{\times}^{r_1+\dots+r_\ell} & \overbrace{\times}^{r_{\ell+1}} & \overbrace{\times}^{n-r_1-\dots-r_{\ell+1}} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

with $r_i \ge 0 \ \forall i$.

Lourenco-Muramatsu-Tsuchiya, 2014 essentially same statement, Liu-P, 2017 stated in terms of reformulations, and short proof

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with $r_i \geq 0 \; \forall i$.

Proof of " $\Leftarrow$ " : Just like in special case

Proof of " $\Rightarrow$ " : (D) asymptotically feasible $\Leftrightarrow$ its Farkas' lemma system

$$A_i \bullet X = 0 \; \forall i, \; C \bullet X = -1, X \succeq 0$$

is infeasible. $+$ Reformulate this system.

Part 3: Asymptotic feasibility, weak infeasibility

- **We can define asymptotic feasibility, weak infeasibility of (P) similarly.**
- **Combining certificates for infeasibility and asymptotic feasibility $\rightarrow$ generating algorithm for all weakly infeasible SDPs. P, Touzov 2022**

Part 4: Ramana's alternative system

Ramana, 1995: A perfect dual for (P) or (D) : no duality gap, always attains optimal value

Gives an exact alternative system for (D) and best complexity results to date.

I will talk about the alternative system for (P).

Part 4: Ramana's alternative system

Motivation: LP infeasibility

$$(P) \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \qquad\qquad \begin{aligned} A^T y &\geq 0 \\ b^T y &= -1 \end{aligned} \quad (ALT\text{-}P)$$

Three desirable properties:

(1) Data of (P) is same as that of $(ALT\text{-}P)$? YES

(2) (P) is infeasible $\Leftrightarrow$ $(ALT\text{-}P)$ is feasible? YES

(3) When feasible, they have poly size solutions ? YES

Part 4: Ramana's alternative system

Motivation: LP infeasibility

$$(LP\text{-}P) \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \qquad\qquad \begin{aligned} A^T y &\geq 0 \\ b^T y &= -1 \end{aligned} \quad (ALT\text{-}P)$$

Three desirable properties:

(1) Data of (P) is same as that of $(ALT\text{-}P)$? YES

(2) (P) is infeasible $\Leftrightarrow$ $(ALT\text{-}P)$ is feasible? YES

(3) When feasible, they have poly size solutions ? YES

Corollary: LP feasibility $\in$ NP $\cap$ co-NP in both Turing and real number model. Known already in the sixties.

Part 4: Ramana's alternative system

Now to SDP

$$\mathcal{A}X := (A_1 \bullet X, \ldots, A_m \bullet X)^\top, \; \mathcal{A}^* y := \sum_{i=1}^m y_i A_i$$

$$\begin{array}{llll} (P) & \mathcal{A}X = b & \mathcal{A}^* y \succeq 0 & (ALT\text{-}P) \\ & X \succeq 0 & b^T y = -1 & \end{array}$$

$(ALT\text{-}P)$ is traditional alternative system

Three desirable properties:

(1) Data of (P) is same as that of $(ALT\text{-}P)$? YES

(2) (P) is infeasible $\Leftrightarrow$ $(ALT\text{-}P)$ is feasible? NO: $\Rightarrow$ may fail.

(3) When feasible, they have poly size solutions ? NO

Part 4: Ramana's alternative system

Now to SDP

$$\mathcal{A}X := (A_1 \bullet X, \ldots, A_m \bullet X)^\top, \ \mathcal{A}^* y := \sum_{i=1}^m y_i A_i$$

$$\begin{array}{lrcl}
(P) & \mathcal{A}X & = & b \\
 & X & \succeq & 0
\end{array}
\qquad\qquad
\begin{array}{rcll}
\mathcal{A}^* y & \succeq & 0 & (ALT\text{-}P) \\
b^T y & = & -1 &
\end{array}$$

$(ALT\text{-}P)$ **is traditional alternative system**

Three desirable properties:

(1) Data of (P) **is same as that of** $(ALT\text{-}P)$ **? YES**

(2) (P) **is infeasible** $\Leftrightarrow$ $(ALT\text{-}P)$ **is feasible? NO**

(3) When feasible, they have poly size solutions ? NO

Corollary: $(ALT\text{-}P)$ **is not useful for complexity**

Part 4: Ramana's alternative system (Ram-Alt-P)

Three desirable properties:

(1) Data of (P) is same as that of (Ram-Alt-P) ? YES

(2) (P) is infeasible $\Leftrightarrow$ (Ram-Alt-P) is feasible? YES

(3) When feasible, they have poly size solutions ? NO

Part 4: Ramana's alternative system (Ram-Alt-P)

Three desirable properties:

(1) Data of (P) is same as that of (Ram-Alt-P) ? YES

(2) (P) is infeasible $\Leftrightarrow$ (Ram-Alt-P) is feasible? YES

(3) When feasible, they have poly size solutions ? NO

Corollary: SDP feasibility

- $\in$ **NP $\cap$ co-NP in real number model.**
- **is not NP complete in Turing model, unless NP = co-NP**
- **After 30 years, still the best result we have.**
- **Very popular in computer science community.**

Ramana's alternative system (Ram-Alt-P)

(P) is infeasible $\Leftrightarrow \exists\ 0 \le k \le n-1:$

$$
\begin{aligned}
U_0 = V_0 &= 0 \\
\left.\begin{aligned}
\mathcal{A}^* y^i &= U_i + V_i \\
b^T y^i &= 0 \\
U_i &\in \mathbb{S}^n_+ \\
V_i &\in \tan(U_{i-1})
\end{aligned}\right\} &\quad \text{for } i \in [k] \\
\left.\begin{aligned}
\mathcal{A}^* y &\in \mathbb{S}^n_+ + \tan(U_k) \\
b^T y &= -1
\end{aligned}\right\} &\quad \text{Like (Alt-P)}
\end{aligned}
$$

is feasible.

Ramana's alternative system (Ram-Alt-P)

(P) is infeasible $\Leftrightarrow \exists\, 0 \le k \le n-1$:

$$\begin{array}{rcl} U_0 = V_0 & = & 0 \\ \left.\begin{array}{rcl} \mathcal{A}^* y^i & = & U_i + V_i \\ b^T y^i & = & 0 \\ U_i & \in & \mathbb{S}^n_+ \\ V_i & \in & \tan\left(U_{i-1}\right) \end{array}\right\} & & \text{for } i \in [k] \\ \left.\begin{array}{rcl} \mathcal{A}^* y & \in & \mathbb{S}^n_+ + \tan\left(U_k\right) \\ b^T y & = & -1 \end{array}\right\} & & \text{Like (Alt-P)} \end{array}$$

is feasible.

- $\tan(U) = \tan(U, \mathbb{S}^n_+) = \left\{ V \in \mathbb{S}^n \,:\, \mathrm{dist}(U \pm \epsilon V, \mathbb{S}^n_+) = o(\epsilon) \right\}$
- $\tan(U)$ is tangent space of $\mathbb{S}^n_+$ at U.

$$\text{E.g.} \quad \tan \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ni \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ since } \begin{pmatrix} 1 & \epsilon \\ \epsilon & \epsilon^2 \end{pmatrix} \succeq 0.$$

Ramana's alternative system (Ram- Alt - P)

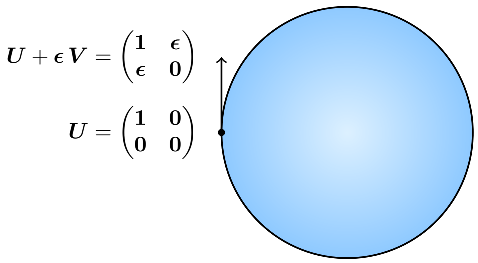
(P) is infeasible $\Leftrightarrow \exists\; 0 \le k \le n-1:$

$$\left.\begin{aligned} U_0 = V_0 &= 0 \end{aligned}\right.$$

$$\left.\begin{aligned} \mathcal{A}^* y^i &= U_i + V_i \\ b^T y^i &= 0 \\ U_i &\in \mathbb{S}^n_+ \\ V_i &\in \tan(U_{i-1}) \end{aligned}\right\} \text{ for } i \in [k]$$

$$\left.\begin{aligned} \mathcal{A}^* y &\in \mathbb{S}^n_+ + \tan(U_k) \\ b^T y &= -1 \end{aligned}\right\} \text{ Like (Alt-P)}$$

is feasible.



The tangent space

Ramana's alternative system (Ram-Alt-P)

(P) is infeasible $\Leftrightarrow \exists\ 0 \leq k \leq n-1:$

$$
\begin{array}{rcll}
U_0 = V_0 & = & 0 & \\
\mathcal{A}^* y^i & = & U_i + V_i & \\
b^T y^i & = & 0 & \\
U_i & \in & \mathbb{S}^n_+ & \\
V_i & \in & \tan(U_{i-1}) & \Bigg\} \text{ for } i \in [k] \\
\mathcal{A}^* y & \in & \mathbb{S}^n_+ + \tan(U_k) & \\
b^T y & = & -1 & \Big\} \text{ Like (Alt-P)}
\end{array}
$$

is feasible.

- Naturally, at least as strong as classical alternative (Alt-P) since $0 \in \tan(U)\ \forall\, U \succeq 0$

Ramana's alternative system (Ram-Alt-P)

(P) is infeasible $\Leftrightarrow \exists\ 0 \le k \le n-1$:

$$
\begin{array}{rcll}
U_0 = V_0 & = & 0 & \\
\mathcal{A}^* y^i & = & U_i + V_i & \\
b^T y^i & = & 0 & \\
U_i & \in & \mathbb{S}^n_+ & \\
V_i & \in & \tan(U_{i-1}) &
\end{array}
\Bigg\} \text{ for } i \in [k]
$$

$$
\left.
\begin{array}{rcl}
\mathcal{A}^* y & \in & \mathbb{S}^n_+ + \tan(U_k) \\
b^T y & = & -1
\end{array}
\right\} \text{ Like (Alt-P)}
$$

is feasible.

- $\tan(U)$ is representable by SDP:
- $\tan(U) = \left\{ W + W^\top : \begin{pmatrix} U & W \\ W^\top & \lambda I \end{pmatrix} \succeq 0\ \exists \lambda \ge 0 \right\}.$

Ramana's alternative system (Ram-Alt-P)

(P) is infeasible $\Leftrightarrow \exists\ 0 \leq k \leq n-1:$

$$
\begin{aligned}
U_0 = V_0 &= 0 \\
\left.\begin{aligned}
\mathcal{A}^* y^i &= U_i + V_i \\
b^T y^i &= 0 \\
U_i &\in \mathbb{S}^n_+ \\
V_i &\in \tan(U_{i-1})
\end{aligned}\right\} &\quad \text{for } i \in [k] \\
\left.\begin{aligned}
\mathcal{A}^* y &\in \mathbb{S}^n_+ + \tan(U_k) \\
b^T y &= -1
\end{aligned}\right\} &\quad \text{Like (Alt-P)}
\end{aligned}
$$

is feasible.

- In words: in (Ram-Alt-P) we decompose each $\mathcal{A}^* y^i$ and $\mathcal{A}^* y$ into a psd part and a tangent space part.
- $\Leftarrow$ is easy
- $\Rightarrow$ is hard.
- We show: proof of $\Rightarrow$ naturally fits into reformulation framework.

Proof sketch of $\Rightarrow$:

$$\begin{aligned} U_0 = V_0 &= 0 \\ \left.\begin{aligned} \mathcal{A}^* y_i &= U_i + V_i \\ b^T y_i &= 0 \\ U_i &\in \mathbb{S}^n_+ \\ V_i &\in \tan(U_{i-1}) \end{aligned}\right\} &\quad \text{for } i \in [k] \\ \left.\begin{aligned} \mathcal{A}^* y &\in \mathbb{S}^n_+ + \tan(U_k) \\ b^T y &= -1 \end{aligned}\right\} &\quad \text{Like (Alt-P)} \end{aligned}$$

Key observation: (Ram-Alt-P) invariant under reformulating (P). (E.g. rotations preserve tangent space)

So we assume infeasible (P) was reformulated into $(P_{\text{ref,infeas}})$.

Proof sketch of $\Rightarrow$:

$$\left.\begin{aligned} U_0 = V_0 &= 0 \\ \mathcal{A}^* y_i &= U_i + V_i \\ b^T y_i &= 0 \\ U_i &\in \mathbb{S}^n_+ \\ V_i &\in \tan(U_{i-1}) \end{aligned}\right\} \text{for } i \in [k]$$

$$\left.\begin{aligned} \mathcal{A}^* y &\in \mathbb{S}^n_+ + \tan(U_k) \\ b^T y &= -1 \end{aligned}\right\} \text{Like (Alt-P)}$$

Key observation: (Ram-Alt-P) invariant under reformulating (P). (E.g. rotations preserve tangent space)

So we assume infeasible (P) was reformulated into $(P_{\text{ref,infeas}})$.

Sketch of proof sketch: incremental structure in both (Ram-Alt-P) and $(P_{\text{ref,infeas}})$.

$\rightarrow$ We will prove: $y_1 := e_1, \ldots, y_k := e_k, y := e_{k+1}$ feasible in (Ram-Alt-P)

Useful fact about tangent spaces

$$\mathbf{tan}\begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \times & \times \\ \times & \mathbf{0} \end{pmatrix}$$

E.g.

$$\mathbf{tan}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \ni \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

Proof of (P) infeasible $\Rightarrow$ (Ram - Alt - P) is feasible

$$\underbrace{\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}}_{A_1} \bullet X = 0$$

$$\underbrace{\begin{pmatrix} \times_{r_1} & \times & \times \\ \times & I_{r_2} & 0 \\ \times & 0 & 0 \end{pmatrix}}_{A_2} \bullet X = 0$$

$$\vdots \qquad (P_{\text{ref,infeas}})$$

$$\underbrace{\begin{pmatrix} \times_{r_1+\cdots+r_k} & \times & \times \\ \times & I_{r_{k+1}} & 0 \\ \times & 0 & 0 \end{pmatrix}}_{A_{k+1}} \bullet X = -1$$

$$\vdots$$

$$X \succeq 0$$

Matrices naturally decompose

$$\underbrace{\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}}_{A_1 = U_1 \succeq 0}$$

$$\underbrace{\begin{pmatrix} \times_{r_1} & \times & \times \\ \times & I_{r_2} & 0 \\ \times & 0 & 0 \end{pmatrix}}_{A_2} = \underbrace{\begin{pmatrix} I_{r_1} & 0 & 0 \\ 0 & I_{r_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{U_2 \succeq 0} + \underbrace{\begin{pmatrix} \times_{r_1} & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}}_{V_2 \in \tan(U_1)}$$

$$\underbrace{\begin{pmatrix} \times_{r_1+r_2} & \times & \times \\ \times & I_{r_3} & 0 \\ \times & 0 & 0 \end{pmatrix}}_{A_3} = \underbrace{\begin{pmatrix} I_{r_1+r_2} & 0 & 0 \\ 0 & I_{r_3} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{U_3 \succeq 0} + \underbrace{\begin{pmatrix} \times_{r_1+r_2} & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}}_{V_3 \in \tan(U_2)} \cdots$$

$\Rightarrow y_1 := e_1, \ldots, y_k := e_k, y := e_{k+1}$ **feasible in (Ram-Alt-P)**

Machinery used

- **At start: subdifferentials, relative interiors, faces, conjugate faces, gauge functions, quadratic modules . . .**
- **More recently, only 2 ingredients:**
  - **Strong duality under Slater condition**
  - **Basic linear algebra**
- **Analogy with LP: in LP we need strong duality+ basic linear algebra.**

Papers

- **P:** Bad semidefinite programs: they all look the same, **2010–SIOPT 2017**
- **Liu–P:** Exact duality in semidefinite programming based on elementary reformulations, **SIOPT 2015**
- **Liu–P:** Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming, **MPA 2017**.
- **P:** Characterizing bad semidefinite programs: normal forms and short proofs, **SIREV 2017**
- **Zhu–P–Tran-Dinh:** Sieve-SDP: a simple facial reduction algorithm to preprocess semidefinite programs, **MPC, 2018**
- **Lourenco-Muramatsu-Tsuchiya:** A structural geometrical analysis of weakly infeasible SDPs, **J. Oper. Res. Soc. Jpn, 2016**
- **Permenter-Parrilo:** Partial facial reduction: simplified, equivalent SDPs via approximations of the PSD cone, **MPA, 2017**
- **Permenter-Friberg-Andersen:** Solving conic optimization problems via self-dual embedding and facial reduction **SIOPT 2017**
- **P:** A Combinatorial Approach to Ramana's Exact Dual for Semidefinite Programming, 2024

On Ramana's dual

- **Ramana, Tuncel, Wolkowicz, 1997, P 2000** other correctness proof, connection to FR
- **de Klerk, Terlaky, Roos 2000** Use of Ramana's dual in self-dual embeddings.
- **P 2013** again, connection to FR, and generalization to conic LPs over nice cones
- **Klep, Schweighofer 2013** exact dual, with analogous properties, based on algebraic geometry
- **Liu-P 2017** Ramana dual for arbitrary conic LPs
- **Lourenço-P 2022** One more correctness proof

Thank you!