A Combinatorial Approach to Ramana's Exact Dual for Semidefinite Programming

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A pair of Semidefinite Programs (SDP)

$$
(P) \quad \begin{array}{ll} \inf_X & C \bullet X \\ s.t. & X \succeq 0 \\ & A_i \bullet X = b_i \, (i \in [m]). \end{array}
\qquad\qquad
\begin{array}{l} \sup_y \; b^T y \\ \sum_{i=1}^m y_i A_i \preceq C \end{array} \quad (D)
$$

Here

- A_i, C are symmetric matrices, $b \in \mathbb{R}^m$.
- $A \preceq B$ means: $B - A$ is positive semidefinite (psd).
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.
- $X \succeq 0 \overset{\text{def}}{\Leftrightarrow}$ all principal subdeterminants are nonnegative.

SDP duality

$$
\begin{array}{llll}
 & \inf_X & C \bullet X & \qquad \sup_y \; b^T y \\
(P) & s.t. & X \succeq 0 & \qquad \sum_{i=1}^m y_i A_i \preceq C \;\; (D) \\
 & & A_i \bullet X = b_i \, (i \in [m]). &
\end{array}
$$

Easy: If X and y are feasible, then $C \bullet X \geq b^T y$.

But:

- unattained optimal values
- positive duality gaps
- Slater's condition helps, but it can fail.
- poly time solvable only under restrictive assumptions

Alternative system to prove infeasibility

$$\begin{array}{llll} & \inf_X & C \bullet X & \\ (P) & s.t. & X \succeq 0 & \\ & & A_i \bullet X = b_i \, (i \in [m]). & \end{array} \qquad \begin{array}{ll} \sum_{i=1}^m y_i A_i \succeq 0 & (ALT\text{-}P) \\ b^\top y = -1 & \end{array}$$

Easy: (P) infeasible $\Leftarrow$ $(ALT\text{-}P)$ feasible

But:

- $\not\Rightarrow$
- Thus $(ALT\text{-}P)$ is not useful for complexity
- not known whether feasibility of (P) is in P

Example

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} \bullet X = \underbrace{0}_{b_1}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A_2} \bullet X = \underbrace{-1}_{b_2}$$

$$X \succeq 0$$

Why infeasible?

- **Suppose** X **feasible** $\Rightarrow x_{11} = 0$
  $\Rightarrow x_{12} = x_{13} = 0$
  $\Rightarrow x_{22} = -1$, **contradiction!**

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$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A_2} \bullet X = \underbrace{-1}_{b_2}$$

$$X \succeq 0$$

But $(ALT\text{-}P)$ is also infeasible: $\nexists y \in \mathbb{R}^2$ s.t.

$$y_1 A_1 + y_2 A_2 \succeq 0, \; y_1 b_1 + y_2 b_2 = -1.$$

Ramana's amazing results from 1995

- **An exact dual of (D) : equal optimal values, and Ramana dual always attains, when finite**
- **Exact infeasibility certificate of (D) : it is feasible $\Leftrightarrow$ (D) infeasible**
- **Data of Ramana dual and Ramana infeasibility certificate is same as data of (D).**

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Corollary: SDP feasibility

- $\in$ **NP$\cap$co-NP in real number model.**
- **is either in $\in$ NP$\cap$co-NP or $\notin$ NP$\cup$co-NP in Turing model.**
- **is not NP complete in Turing model, unless NP$=$co-NP**

After 30 years, still the best result we have.

A lot of followup work

- **335 citations**
- **Other correctness proofs, connection to FR: Ramana, Tuncel, Wolkowicz 1997; P 2000**
- **Luo, Sturm, Zhang 2000: another correctness proof**
- **De Klerk, Terlaky, Roos 2000: Use of Ramana's dual in self-dual embeddings.**
- **P 2000, 2013: generalization to conic LPs over nice cones .**
- **Klep, Schweighofer 2013: exact dual, with analogous properties, based on algebraic geometry**
- **Liu-P 2017: Ramana dual for arbitrary conic LPs**
- **Lourenço-P: 2022 One more correctness proof**

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- **Lourenço-P: 2022 One more correctness proof**
- **Very popular in computer science community.**
- **Fairly long/technical proofs: subdifferentials, relative interiors, faces, conjugate faces, gauge functions, quadratic modules ...**
- **Structure of feasible solutions of Ramana's dual and alternative system is not yet completely understood**

Plan

(1) State a suitable variant of Ramana alternative system of (P)

(2) Recall reformulations from Liu-P, 2015, SIOPT

(3) Give elementary proof of alternative system's correctness

Plan

(1) State a suitable variant of Ramana alternative system of (P)

(2) Recall reformulations from Liu-P, 2015, SIOPT

(3) Give elementary proof of alternative system's correctness

Only use two ingredients:

- Strong duality under Slater's condition
- Basic linear algebra

(4) Analogous elementary proof for Ramana dual

(5) More in depth analysis than previously available: characterize not just optimal value, but also feasible set.

Notation:

$$\mathcal{A}X := (A_1 \bullet X, \ldots, A_m \bullet X)^\top, \ \mathcal{A}^* y := \sum_{i=1}^m y_i A_i$$

Ramana's alternative system (Ram-Alt-P)

(P) is infeasible $\Leftrightarrow \exists\ 0 \leq k \leq n-1$:

$$
\begin{aligned}
U_0 = V_0 &= 0 \\
\left.\begin{aligned}
\mathcal{A}^* y^i &= U_i + V_i \\
b^T y^i &= 0 \\
U_i &\in \mathbb{S}^n_+ \\
V_i &\in \tan(U_{i-1})
\end{aligned}\right\} &\quad \text{for } i \in [k] \\
\left.\begin{aligned}
\mathcal{A}^* y &\in \mathbb{S}^n_+ + \tan(U_k) \\
b^T y &= -1
\end{aligned}\right\} &\quad \text{Like (Alt-P)}
\end{aligned}
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is feasible.

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is feasible.

- $\tan(U) = \tan(U, \mathbb{S}^n_+) = \left\{ V \in \mathbb{S}^n : \operatorname{dist}(U \pm \epsilon V, \mathbb{S}^n_+) = o(\epsilon) \right\}$
- $\tan(U)$ is tangent space of $\mathbb{S}^n_+$ at U.

$$\text{E.g.} \quad \tan \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ni \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ since } \begin{pmatrix} 1 & \epsilon \\ \epsilon & \epsilon^2 \end{pmatrix} \succeq 0.$$

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$$

is feasible.

- $\tan(U)$ is representable by SDP:
- $\tan(U) = \left\{ W + W^\top : \begin{pmatrix} U & W \\ W^\top & \lambda I \end{pmatrix} \succeq 0 \ \exists \lambda \geq 0 \right\}.$

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is feasible.

- Naturally, at least as strong as classical alternative (Alt-P) since $0 \in \tan(U)\ \forall\ U \succeq 0$

Ramana's alternative system (Ram-Alt-P)

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is feasible.

- In words: in (Ram-Alt-P) we decompose each $\mathcal{A}^* y^i$ and $\mathcal{A}^* y$ into a psd part and a tangent space part.
- $\Leftarrow$ is easy
- $\Rightarrow$ is hard.

Reformulations

$$
(P) \qquad \begin{array}{rl} \inf_X & C \bullet X \\ s.t. & X \succeq 0 \\ & A_i \bullet X = b_i \, (i \in [m]). \end{array}
$$

We obtain a reformulation of (P) by a sequence of the following:

- Elementary row operations on the equations of (P)
- For a V invertible matrix

$$A_i \leftarrow V^T A_i V \; (i \in [m]), C \leftarrow V^T C V.$$

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A'_i \bullet X &= 0 \ (i = 1, \ldots, k) \\ A'_{k+1} \bullet X &= -1 \qquad\qquad\qquad (\mathrm{P_{ref}}) \\ &\vdots \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, **and for** $i = 1, \ldots, k+1$

$$A'_1 = \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix}, \ A'_i = \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

with $r_i \geq 0 \ \forall i$.

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$
\begin{aligned}
A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\
A_{k+1}' \bullet X &= -1 \qquad\qquad (\mathrm{P_{ref}}) \\
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$$

with $r_i \geq 0 \ \forall i$. Liu-P, 2015

Proof of " $\Leftarrow$ " : Suppose X feasible in $(\mathrm{P_{ref}})$

$\Rightarrow$ first r_1 rows of X are 0

$\ldots$

$\Rightarrow$ first $r_1 + \ldots + r_k$ rows of X are 0

$\Rightarrow$ trace of diag. block of X is -1, contradiction!

Proof of " $\Rightarrow$ " : Also not hard P-Touzov, 2022

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

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$$

with $r_i \geq 0 \ \forall i$.

Standard form of empty spectrahedra.

Example, with $k = 1$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A'_1} \bullet X = 0$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A'_2} \bullet X = -1$$

$$X \succeq 0$$

Back to Ramana's alternative system (Ram-Alt-P)

(P) is infeasible $\Leftrightarrow \exists\ 0 \leq k \leq n-1$:

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\begin{array}{rcl}
U_0 = V_0 & = & 0 \\
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U_i & \in & \mathbb{S}^n_+ \\
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\end{array}\right\} & & \text{for } i \in [k] \\
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$$

is feasible. Key observation: (Ram-Alt-P) invariant under reformulating (P).

Back to Ramana's alternative system (Ram-Alt-P)

(P) is infeasible $\Leftrightarrow \exists\ 0 \le k \le n-1$:

$$\left.\begin{aligned} U_0 = V_0 &= 0 \\ \mathcal{A}^* y^i &= U_i + V_i \\ b^T y^i &= 0 \\ U_i &\in \mathbb{S}^n_+ \\ V_i &\in \tan(U_{i-1}) \end{aligned}\right\} \text{ for } i \in [k]$$

$$\left.\begin{aligned} \mathcal{A}^* y &\in \mathbb{S}^n_+ + \tan(U_k) \\ b^T y &= -1 \end{aligned}\right\} \text{ Like (Alt-P)}$$

is feasible. Key observation: (Ram - Alt - P) invariant under reformulating (P).

Why? (P) constraints are $\mathcal{A}X = b$. Reformulations

- replace $\mathcal{A} \leftarrow M\mathcal{A}$, $b \leftarrow Mb$, so $\mathcal{A}^*$ by $\mathcal{A}^* M^*$ for M invertible.
- do rotations, which preserve tangent spaces.

So we assume infeasible (P) was reformulated into (P_{ref}).

Useful fact about tangent spaces

$$\mathbf{tan}\begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \times & \times \\ \times & \mathbf{0} \end{pmatrix}$$

E.g.

$$\mathbf{tan}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \ni \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

Matrices in $(\mathrm{P_{ref}})$ naturally decompose

$$\underbrace{\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}}_{A_1 = U_1 \succeq 0}$$

$$\underbrace{\begin{pmatrix} \times_{r_1} & \times & \times \\ \times & I_{r_2} & 0 \\ \times & 0 & 0 \end{pmatrix}}_{A_2} = \underbrace{\begin{pmatrix} I_{r_1} & 0 & 0 \\ 0 & I_{r_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{U_2 \succeq 0} + \underbrace{\begin{pmatrix} \times_{r_1} & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}}_{V_2 \in \tan(U_1)}$$

$$\underbrace{\begin{pmatrix} \times_{r_1+r_2} & \times & \times \\ \times & I_{r_3} & 0 \\ \times & 0 & 0 \end{pmatrix}}_{A_3} = \underbrace{\begin{pmatrix} I_{r_1+r_2} & 0 & 0 \\ 0 & I_{r_3} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{U_3 \succeq 0} + \underbrace{\begin{pmatrix} \times_{r_1+r_2} & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}}_{V_3 \in \tan(U_2)} \cdots$$

$\Rightarrow y^1 := e_1, \ldots, y^k := e_k, y := e_{k+1}$ feasible in (Ram-Alt-P)

Part 2: Ramana's dual (Ram-D)

Recall our SDP pair:

$$
\begin{array}{llll}
 & \inf_X \ C \bullet X & & \sup_y \ b^T y \\
(P) & s.t. \ X \succeq 0 & & \sum_{i=1}^m y_i A_i \preceq C \ (D) \\
 & A_i \bullet X = b_i \, (i \in [m]). & &
\end{array}
$$

Unattained optimal values, positive duality gaps . . . so we want a better dual.

Prep: the strong dual

Suppose a max rank feasible solution (P) is

$$X = Q \begin{pmatrix} 0 & 0 \\ 0 & \underbrace{\Lambda}_{r} \end{pmatrix} Q^\top, \text{ with } Q \text{ orthonormal, } \Lambda \text{ positive definite.}$$

Then the optimal values of (P) and

$$\begin{aligned} &\sup\ b^\top y \\ &s.t.\ C - \textstyle\sum_{i=1}^m y_i A_i \in Q \begin{pmatrix} \times & \times \\ \times & \oplus_r \end{pmatrix} Q^\top \quad (\mathrm{D_{strong}}) \end{aligned}$$

agree, and latter is attained when finite.

So, a good strong dual.

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agree, and latter is attained when finite.

So, a good strong dual.

Catch: in general, we do not know such an X! So, not an explicit SDP.

Ramana's dual $(\mathrm{D_{Ram}})$

$$
\begin{aligned}
\sup \quad & b^\top y \\
\text{s.t.} \quad & C - \mathcal{A}^* y \in \mathcal{S}^n_+ + \tan(U_{n-1}) \\
& U_0 = V_0 = 0 \\
& \left.\begin{aligned}
\mathcal{A}^* y^i &= U_i + V_i \\
b^\top y^i &= 0 \\
U_i &\in \mathcal{S}^n_+ \\
V_i &\in \tan(U_{i-1})
\end{aligned}\right\} \quad \text{for } i \in [n-1]
\end{aligned}
$$

Explicit SDP

Ramana's dual $(\mathrm{D}_{\mathrm{Ram}})$

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\sup & b^\top y \\
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\end{array}
$$

Explicit SDP

Theorem

$$y \text{ is feasible in } (\mathrm{D}_{\mathrm{strong}})$$

$$\Leftrightarrow$$

$$y \text{ is feasible in } (\mathrm{D}_{\mathrm{Ram}}) \text{ with some } \{y^i, U_i, V_i\}$$

Ramana's dual $(\mathrm{D}_{\mathrm{Ram}})$

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\begin{array}{ll}
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\end{array}
$$

Feasible set of $(\mathrm{D}_{\mathrm{strong}})$ is lifted into feasible set of $(\mathrm{D}_{\mathrm{Ram}})$

The latter is a lift, or extended formulation of the former.

Lifts, extended formulations: Balas; Conforti-Cornuejols-Zambelli; Gouveia-Parrilo-Thomas; ...

Corollary Strong duality between (P) and $(\mathrm{D}_{\mathrm{Ram}})$

Machinery used

- **At start: subdifferentials, relative interiors, faces, conjugate faces, gauge functions, quadratic modules ...**
- **More recently, only 2 ingredients:**
  - **Strong duality under Slater condition**
  - **Basic linear algebra**
- **Analogy with LP: in LP we need strong duality+ basic linear algebra.**

Thank you!