How do Exponential Size Solutions Arise in Semidefinite Programming?

Gábor Pataki

UNC Chapel Hill

Joint work with Alex Touzov

Foundations of Computational Mathematics, Paris

June 17, 2023

Semidefinite Programing (SDP) feasibility

$\exists ? \boldsymbol{x}$ s.t.

$$\sum_{i=1}^{m} \boldsymbol{x}_i \boldsymbol{A}_i + \boldsymbol{B} \succeq \mathbf{0} \qquad (\mathbf{SDP})$$

Here

- $\boldsymbol{A}_i, \boldsymbol{B}$ are symmetric matrices,
- $\boldsymbol{S} \succeq \mathbf{0}$ means that $\boldsymbol{S}$ is symmetric positive semidefinite (psd).

Semidefinite Programing (SDP) feasibility

$\exists ? x$ s.t.

$$\sum_{i=1}^{m} x_i A_i + B \succeq 0 \qquad (\mathbf{SDP})$$

Here

- A_i, B are symmetric matrices,
- $S \succeq 0$ means that S is symmetric positive semidefinite (psd).

Terminology: size of a number

= number of bits needed to describe it.

Example: Size of $p \in \mathbb{Z}$ is $\lceil \log(|p| + 1) \rceil + 1$

Khachiyan SDP with exponential size solutions

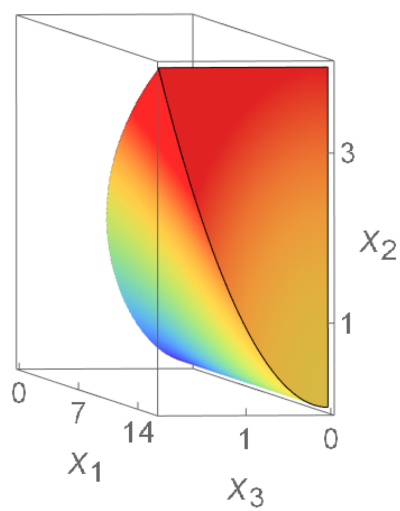
- $x_1 \geq x_2^2,\ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2,\ x_m \geq 2.$ (**Khachiyan**)
- x feasible $\Rightarrow x_1 \geq 2^{2^{m-1}}$.
- $\Rightarrow$ Size of $x \geq \log 2^{2^{m-1}} = 2^{m-1}$.

Khachiyan SDP with exponential size solutions

- $x_1 \geq x_2^2,\ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2,\ x_m \geq 2.$ (Khachiyan)
- x feasible $\Rightarrow x_1 \geq 2^{2^{m-1}}.$
- $\Rightarrow$ Size of $x \geq \log 2^{2^{m-1}} = 2^{m-1}.$
- Can be written as SDP:

$$x_i \geq x_{i+1}^2 \;\Leftrightarrow\; \begin{pmatrix} x_i & x_{i+1} \\ x_{i+1} & 1 \end{pmatrix} \succeq 0 \;\forall i.$$

- Picture:



This is not just about the existence of exponential size solutions!

$$x_1 \geq x_2^2,\ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2,\ x_m \geq 2.$$

- **Exponential size solutions exist when the feasible set is unbounded (even in LP).**

This is not just about the existence of exponential size solutions!

$$x_1 \geq x_2^2,\ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2,\ x_m \geq 2.$$

- Exponential size solutions exist when the feasible set is unbounded (even in LP).
- In (Khachiyan) all solutions must have exponential size.
- Key point: hierarchy among the variables.

Is (SDP) feasibility in P?

- Major open problem
- Open even for quadratic constraints

Is (SDP) feasibility in P?

- **Major open problem**
- **Open even for quadratic constraints**
- **Exponential size solutions are a major obstacle**
- **How to prove in polynomial time that a possibly exponential size solution exists?**

Questions

(1) **Are SDPs with such large solutions common?**

- **Maybe not . . . we rarely see them; and large solutions are easy to destroy even in (Khachiyan):**
  - **Replace $x_m \geq 2$ by $x_m \geq 2 + x_{m+1}$;**
  - **Replace x by Gx where G is some random invertible matrix.**

Questions

(1) **Are SDPs with such large solutions common?**

- **Maybe not . . . we rarely see them; and large solutions are easy to destroy even in (Khachiyan):**
  - **Replace $x_m \geq 2$ by $x_m \geq 2 + x_{m+1}$;**
  - **Replace x by Gx where G is some random invertible matrix.**

(2) **Can we represent such large solutions in polynomial space?**

- **Maybe yes . . . (Khachiyan) gives hope: no need to write out $2^{2^{m-1}}$ to prove that $x_1 = 2^{2^{m-1}}$ is feasible.**

  We can just do a symbolic computation!

Questions

(1) **Are SDPs with such large solutions common?**

- **Maybe not . . . we rarely see them; and large solutions are easy to destroy even in (Khachiyan):**
  - **Replace $x_m \geq 2$ by $x_m \geq 2 + x_{m+1}$;**
  - **Replace x by Gx where G is some random invertible matrix.**

(2) **Can we represent such large solutions in polynomial space?**

- **Maybe yes . . . (Khachiyan) gives hope: no need to write out $2^{2^{m-1}}$ to prove that $x_1 = 2^{2^{m-1}}$ is feasible.**

  We can just do a symbolic computation!

However: we give a partial "yes" answer to both (1) and (2)

- **Background:**

$$\boldsymbol{k} := \text{singularity degree of } \{\, \boldsymbol{Y} \succeq 0 : \boldsymbol{A}_i \bullet \boldsymbol{Y} = 0 \, \forall \boldsymbol{i} \,\}.$$

- **We assume (SDP) is strictly feasible, i.e., $\exists \boldsymbol{x}$ s.t.**

$$\sum_{i=1}^{m} \boldsymbol{x}_i \boldsymbol{A}_i + \boldsymbol{B} \succ 0.$$

Theorem 1 (Informal)

$\exists$ an invertible matrix M s.t. the linear change of variables $x \leftarrow Mx$ transforms (SDP) into (SDP') with the following properties:

If x strictly feasible in (SDP') and x_k is large enough, then

$$x_1 \geq d_2 x_2^{\alpha_2},\ x_2 \geq d_3 x_3^{\alpha_3}, \ldots,\ x_{k-1} \geq d_k x_k^{\alpha_k}$$

Theorem 1 (Informal)

$\exists$ **an invertible matrix** M **s.t. the linear change of variables** $x \leftarrow Mx$ **transforms** (**SDP**) **into** (**SDP'**) **with the following properties:**

If x **strictly feasible in** (**SDP'**) **and** x_k **is large enough, then**

$$x_1 \geq d_2 x_2^{\alpha_2},\ x_2 \geq d_3 x_3^{\alpha_3}, \ldots,\ x_{k-1} \geq d_k x_k^{\alpha_k}$$

where

$$2 \geq \alpha_2 \geq \frac{k}{k-1}, 2 \geq \alpha_3 \geq \frac{k-1}{k-2}, \ldots,\ 2 \geq \alpha_k \geq 2.$$

Theorem 1 (Informal)

$\exists$ an invertible matrix M s.t. the linear change of variables $x \leftarrow Mx$ transforms (SDP) into (SDP') with the following properties:

If x strictly feasible in (SDP') and x_k is large enough, then

$$x_1 \geq d_2 x_2^{\alpha_2},\ x_2 \geq d_3 x_3^{\alpha_3}, \ldots,\ x_{k-1} \geq d_k x_k^{\alpha_k}$$

where

$$2 \geq \alpha_2 \geq \frac{k}{k-1}, 2 \geq \alpha_3 \geq \frac{k-1}{k-2}, \ldots,\ 2 \geq \alpha_k \geq 2.$$

The d_j and α_j are constants that depend on the A_i, on B and $x_{k+1}, \ldots, x_m$ that we consider fixed.

Khachiyan type hierarchy in all strictly feasible SDPs.

Theorem 1 (Informal)

$\exists$ an invertible matrix M s.t. the linear change of variables $x \leftarrow Mx$ transforms (SDP) into (SDP') with the following properties:

If x strictly feasible in (SDP') and x_k is large enough, then

$$x_1 \geq d_2 x_2^{\alpha_2},\ x_2 \geq d_3 x_3^{\alpha_3}, \dots,\ x_{k-1} \geq d_k x_k^{\alpha_k}$$

where

$$2 \geq \alpha_2 \geq \frac{k}{k-1}, 2 \geq \alpha_3 \geq \frac{k-1}{k-2}, \dots, 2 \geq \alpha_k \geq 2.$$

The d_j and α_j are constants that depend on the A_i, on B and $x_{k+1}, \dots, x_m$ that we consider fixed.

Khachiyan type hierarchy in all strictly feasible SDPs.

Assumptions are minimal.

Worst case example: Khachiyan SDP

$$\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & & x_3 \\ & & x_3 & & x_4 \\ & & & x_4 & \\ x_2 & x_3 & x_4 & & 1 \end{pmatrix} \succeq 0$$

Worst case example: Khachiyan SDP

$$\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & & x_3 \\ & & x_3 & & x_4 \\ & & & x_4 & \\ x_2 & x_3 & x_4 & & 1 \end{pmatrix} \succeq 0$$

- Subdeterminant with three red corners $\Rightarrow x_1 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

Worst case example: Khachiyan SDP

$$\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & & x_3 \\ & & x_3 & & x_4 \\ & & & x_4 & \\ x_2 & x_3 & x_4 & & 1 \end{pmatrix} \succeq 0$$

- Subdeterminant with three red corners $\Rightarrow x_1 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

Exponents are maximal.

Best case example: “Mild” SDP

$$\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix} \succeq 0$$

Best case example: “Mild” SDP

$$\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix} \succeq 0$$

- Subdeterminant with three red corners $\Rightarrow x_1 x_3 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 x_4 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

Best case example: “Mild” SDP

$$\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix} \succeq 0$$

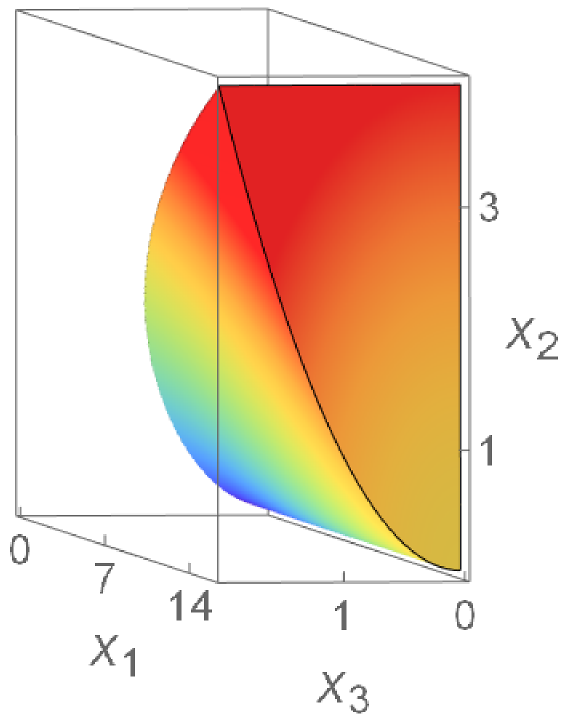
- Subdeterminant with three red corners $\Rightarrow x_1 x_3 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 x_4 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

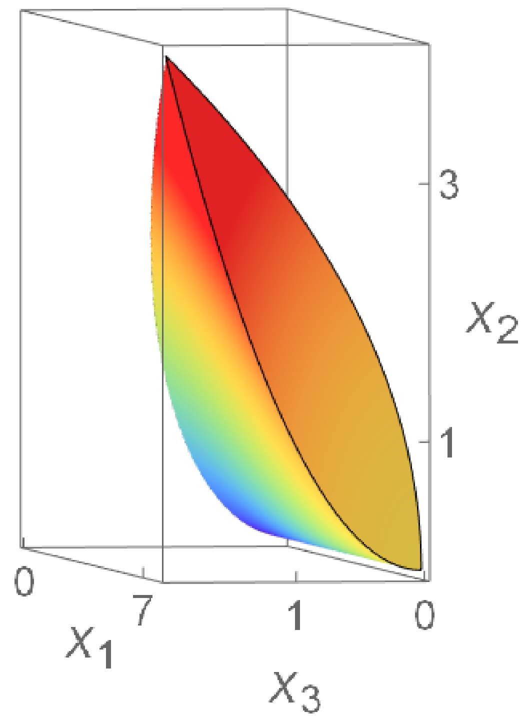
From these we derive:

$$\mathrm{x}_1 \geq \mathrm{x}_2^{4/3},\ \mathrm{x}_2 \geq \mathrm{x}_3^{3/2},\ \mathrm{x}_3 \geq \mathrm{x}_4^2$$

Exponents are minimal.

Khachiyan vs Mild





- **Three variables, $2 \geq x_3 \geq 0$ (normalization)**

Change of variables: (SDP) into (SDP')

The reformulated SDP looks like

$$x_1 \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{r_1} & \overbrace{\quad}^{n-r_1} \end{array} \\ \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \quad + \sum_{i=2}^{k} x_i \begin{array}{c} \begin{array}{ccc} \overbrace{\qquad}^{r_1+\ldots+r_{i-1}} & \overbrace{\quad}^{r_i} & \overbrace{\qquad}^{n-r_1-\ldots-r_i} \end{array} \\ \begin{pmatrix} \times & \times & \times \\ \times & \boldsymbol{I} & \mathbf{0} \\ \times & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}$$

$$+ \sum_{i=k+1}^{m} x_i A_i' + B' \succeq 0 \qquad \text{(SDP')}$$

with $r_1, \ldots, r_k > 0$.

Change of variables: (SDP) into (SDP')

The reformulated SDP looks like

$$x_1 \begin{array}{c} \begin{array}{cc} \overbrace{}^{r_1} & \overbrace{}^{n-r_1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array} + \sum_{i=2}^k x_i \begin{array}{c} \begin{array}{ccc} \overbrace{}^{r_1+\ldots+r_{i-1}} & \overbrace{}^{r_i} & \overbrace{}^{n-r_1-\ldots-r_i} \end{array} \\ \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \end{array}$$

$$+ \sum_{i=k+1}^m x_i A_i' + B' \succeq 0 \qquad \text{(SDP')}$$

with $r_1, \ldots, r_k > 0$.

To get this reformulation, we used

(1) linear change of variables $x \leftarrow Mx$;

(2) a similarity transformation $T^\top ()T$.

(3) Background: facial reduction, reformulations: Borwein-Wolkowicz, Waki-Muramatsu, P, Liu-P, ...

From (SDP') to inequalities $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$

Formula for the α_{j+1} :

$$\alpha_{j+1} = \begin{cases} 2 - \dfrac{1}{\alpha_{j+2} \ldots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\ 2 & \text{if } t_{j+1} = k+1 \end{cases}$$

for $j = 1, \ldots, k-1$.

From (SDP') to inequalities $x_j \geq \mathrm{const} \cdot x_{j+1}^{\alpha_{j+1}}$

Formula for the α_{j+1} :

$$\alpha_{j+1} = \begin{cases} 2 - \dfrac{1}{\alpha_{j+2} \dots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\ 2 & \text{if } t_{j+1} = k+1 \end{cases}$$

for $j = 1, \dots, k-1$.

Similar to continued fractions.

From (SDP') to inequalities $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$

Formula for the α_{j+1} :

$$\alpha_{j+1} = \begin{cases} 2 - \dfrac{1}{\alpha_{j+2} \dots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\ 2 & \text{if } t_{j+1} = k+1 \end{cases}$$

for $j = 1, \dots, k-1$.

Similar to continued fractions.

Here

$$t_{j+1} = \text{index of a rightmost block with } x_{j+1}$$

Shift x_{j+1} to right $\Rightarrow$ t_{j+1} **increases.**

$\Rightarrow$ α_{j+1} **increases.**

Example

$$\underbrace{\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha = (4/3,\ 3/2,\ 2)}$$

Example

$$\underbrace{\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha = (4/3,\ 3/2,\ 2)} \rightarrow \underbrace{\begin{pmatrix} x_1 & & & x_2 & \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ x_2 & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha = (5/3,\ 3/2,\ 2)}$$

Example

$$\underbrace{\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha = (4/3,\ 3/2,\ 2)} \rightarrow \underbrace{\begin{pmatrix} x_1 & & & x_2 & \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ x_2 & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha = (5/3,\ 3/2,\ 2)} \rightarrow \underbrace{\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ x_2 & & x_4 & & 1 \end{pmatrix}}_{\alpha = (2,\ 3/2,\ 2)}$$

Example

$$\underbrace{\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(4/3,\ 3/2,\ 2)} \to \underbrace{\begin{pmatrix} x_1 & & & x_2 & \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ x_2 & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(5/3,\ 3/2,\ 2)} \to \underbrace{\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ x_2 & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(2,\ 3/2,\ 2)}$$

In other words

$$x_1 \geq x_2^{4/3} \quad \to \quad x_1 \geq x_2^{5/3} \quad \to \quad x_1 \geq x_2^2$$

Do we need the change of variables $x \leftarrow Mx$?

- **In general, yes: such an operation may mess up even** (**Khachiyan**).
- **So, we may need such an operation $x \leftarrow M^{-1}x$ to unmess it.**
- **But, sometimes we don't.**

Do we need the change of variables $x \leftarrow Mx$?

- **In general, yes: such an operation may mess up even (Khachiyan).**
- **So, we may need such an operation $x \leftarrow M^{-1}x$ to unmess it.**
- **But, sometimes we don't! Many sum-of-squares SDPs are in the form of (SDP') with no change of variables.**

Do we need the change of variables $x \leftarrow Mx$?

- **In general, yes: such an operation may mess up even (Khachiyan).**
- **So, we may need such an operation $x \leftarrow M^{-1}x$ to unmess it.**
- **But, sometimes we don't! Many sum-of-squares SDPs are in the form of (SDP') with no change of variables.**
- **Ex 1: Minimize univariate polynomial by SDP.**

  In dual SDP: $y_{2n} \geq y_{2n-2}^{1+1/(n-1)}, \; y_{2n-2} \geq y_{2n-4}^{1+1/(n-2)}, \ldots$

  $\Rightarrow y_{2n} \geq y_2^n$

Do we need the change of variables $x \leftarrow Mx$?

- In general, yes: such an operation may mess up even (Khachiyan).
- So, we may need such an operation $x \leftarrow M^{-1}x$ to unmess it.
- But, sometimes we don't! Many sum-of-squares SDPs are in the form of (SDP') with no change of variables.
- Ex 1: Minimize univariate polynomial by SDP.

  In dual SDP: $y_{2n} \geq y_{2n-2}^{1+1/(n-1)}$, $y_{2n-2} \geq y_{2n-4}^{1+1/(n-2)}, \ldots$
  $\Rightarrow y_{2n} \geq y_2^n$
- Ex 2: O' Donnell, 2017 SDP to certify

$$\begin{aligned} x_1 + \cdots + x_n - 2y_1 &\geq 0 \\ s.t.\ x_i &\in \{0,1\} \forall i, \\ y_i &= 0 \, \forall i. \end{aligned} \tag{1}$$

  – In SDP: $u_1 \geq u_2^2$, $u_2 \geq u_3^2, \ldots$

How to certify exponential size solutions in polynomial space ?

Recall reformulated problem:

$$x_1 \begin{array}{c} \begin{array}{cc} \overbrace{}^{r_1} & \overbrace{}^{n-r_1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array} + \sum_{i=2}^{k} x_i \begin{array}{c} \begin{array}{ccc} \overbrace{}^{r_1+\ldots+r_{i-1}} & \overbrace{}^{r_i} & \overbrace{}^{n-r_1-\ldots-r_i} \end{array} \\ \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \end{array}$$

$$+ \sum_{i=k+1}^{m} x_i A_i' + B' \succeq 0 \qquad (\text{SDP'})$$

Suppose we have $x_{k+1}, \ldots, x_m$ s.t. $\exists\, x_1, \ldots, x_k$ so this problem is strictly feasible.

Recall that $x_1, \ldots, x_k$ are "large."

How to certify exponential size solutions in polynomial space ?

Recall reformulated problem:

$$x_1 \begin{pmatrix} \overbrace{\boldsymbol{I}}^{r_1} & \overbrace{\mathbf{0}}^{n-r_1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & \boldsymbol{I} & \mathbf{0} \\ \times & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$+ \sum_{i=k+1}^{m} x_i A_i' + B' \succeq 0 \qquad \text{(SDP')}$$

Suppose we have $x_{k+1}, \ldots, x_m$ **s.t.** $\exists\, x_1, \ldots, x_k$ **so this problem is strictly feasible.**

Recall that $x_1, \ldots, x_k$ **are "large."**

We can prove that $x_1, \ldots, x_k$ **exist, without computing them!**

Proving that $x_1, \ldots, x_k$ exist, without computing them

Start with $Z := \sum_{i=k+1}^{m} x_i A_i' + B'$.

Symbolically add $x_k A_k'$, $x_{k-1} A_{k-1}', \ldots$ **to make larger and larger lower right corners positive definite.**

Proving that $x_1, \ldots, x_k$ exist, without computing them

Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.

Symbolically add $x_k A'_k,\ x_{k-1} A'_{k-1}, \ldots$ to make larger and larger lower right corners positive definite.

$$\underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \underbrace{\boxed{+}}_{\succ 0} \end{pmatrix}}_{Z}$$

Proving that $x_1, \ldots, x_k$ exist, without computing them

Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.

Symbolically add $x_k A'_k,\ x_{k-1} A'_{k-1}, \ldots$ **to make larger and larger lower right corners positive definite.**

$$\underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \underbrace{\boxed{+}}_{\succ 0} \end{pmatrix}}_{Z} \quad \xmapsto[x_k \gg 0]{+x_k A'_k}$$

Proving that $x_1, \ldots, x_k$ exist, without computing them

Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.

Symbolically add $x_k A'_k,\ x_{k-1} A'_{k-1}, \ldots$ **to make larger and larger lower right corners positive definite.**

$$\underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \underbrace{\boxed{+}}_{\succ 0} \end{pmatrix}}_{Z} \quad \xmapsto[x_k \gg 0]{+x_k A'_k} \quad \underbrace{\begin{pmatrix} \begin{matrix} \times & \times & \times & \times \\ \times & \times & \times & \times \end{matrix} \\ \begin{matrix} \times & \times \\ \times & \times \end{matrix} \; \underbrace{\boxed{\begin{matrix} + & \times \\ \times & + \end{matrix}}}_{\succ 0} \end{pmatrix}}_{x_k A'_k + Z}$$

Proving that $x_1, \ldots, x_k$ exist, without computing them

Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.

Symbolically add $x_k A'_k,\ x_{k-1} A'_{k-1}, \ldots$ **to make larger and larger lower right corners positive definite.**

$$\underbrace{\left(\begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \underbrace{\boxed{+}}_{\succ 0} \end{array}\right)}_{Z} \quad \overset{+x_k A'_k}{\underset{x_k \gg 0}{\longmapsto}} \quad \underbrace{\left(\begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \boxed{+} & \boxed{\times} \\ \times & \times & \multicolumn{2}{c}{\underbrace{\boxed{\times} \quad \boxed{+}}_{\succ 0}} \end{array}\right)}_{x_k A'_k + Z} \quad \overset{+x_{k-1} A'_{k-1}}{\underset{x_{k-1} \gg 0}{\longmapsto}}$$

Proving that $x_1, \ldots, x_k$ exist, without computing them

Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.

Symbolically add $x_k A'_k,\ x_{k-1} A'_{k-1}, \ldots$ **to make larger and larger lower right corners positive definite.**

$$\underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \underbrace{+}_{\succ 0} \end{pmatrix}}_{Z} \xmapsto[x_k \gg 0]{+x_k A'_k} \underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & + & \times \\ \times & \times & \times & + \end{pmatrix}}_{x_k A'_k + Z} \xmapsto[x_{k-1} \gg 0]{+x_{k-1} A'_{k-1}} \underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & + & \times & \times \\ \times & \times & + & \times \\ \times & \times & \times & + \end{pmatrix}}_{x_{k-1} A'_{k-1} + x_k A'_k + Z}$$

Proving that $x_1, \ldots, x_k$ exist, without computing them

Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.

Symbolically add $x_k A'_k,\ x_{k-1} A'_{k-1}, \ldots$ **to make larger and larger lower right corners positive definite.**

$$\underbrace{\underset{\succ 0}{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & + \end{pmatrix}}}_{Z} \quad \xmapsto[x_k \gg 0]{+x_k A'_k} \quad \underbrace{\underset{\succ 0}{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & + & \times \\ \times & \times & \times & + \end{pmatrix}}}_{x_k A'_k + Z} \quad \xmapsto[x_{k-1} \gg 0]{+x_{k-1} A'_{k-1}} \quad \underbrace{\underset{\succ 0}{\begin{pmatrix} \times & \times & \times & \times \\ \times & + & \times & \times \\ \times & \times & + & \times \\ \times & \times & \times & + \end{pmatrix}}}_{x_{k-1} A'_{k-1} + x_k A'_k + Z} \quad \xmapsto[x_{k-2} \gg 0]{+x_{k-2} A'_{k-2}} \quad \cdots$$

Proving that $x_1, \ldots, x_k$ exist, without computing them

Start with $Z := \sum_{i=k+1}^{m} x_i A_i' + B'$.

Symbolically add $x_k A_k',\ x_{k-1} A_{k-1}', \ldots$ **to make larger and larger lower right corners positive definite.**

$$\underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \underbrace{+}_{\succ 0} \end{pmatrix}}_{Z} \xrightarrow[x_k \gg 0]{+x_k A_k'} \underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & + & \times \\ \times & \times & \times & + \end{pmatrix}}_{x_k A_k' + Z} \xrightarrow[x_{k-1} \gg 0]{+x_{k-1} A_{k-1}'} \underbrace{\begin{pmatrix} \times & \times & \times & \times \\ \times & + & \times & \times \\ \times & \times & + & \times \\ \times & \times & \times & + \end{pmatrix}}_{x_{k-1} A_{k-1}' + x_k A_k' + Z} \xrightarrow[x_{k-2} \gg 0]{+x_{k-2} A_{k-2}'} \cdots$$

Just like in (**Khachiyan**).

Also inspiration: Lourenço-Muramatsu-Tsuchiya: A structural geometrical analysis of weakly infeasible SDPs

Conclusion

- **Exponential size solutions in SDP, going back to famous Khachiyan example.**
- **Khachiyan type hierarchy among leading variables in every strictly feasible SDP (after linear change of variables)**
- **Formulas to compute the exponents (like continued fractions)**
- **Connection to Fourier-Motzkin elimination (pls see paper)**

Conclusion

- **Exponential size solutions in SDP, going back to famous Khachiyan example.**
- **Khachiyan type hierarchy among leading variables in every strictly feasible SDP (after linear change of variables)**
- **Formulas to compute the exponents (like continued fractions)**
- **Partial answer to: how to represent exponential size solutions in polynomial space?**
- **Every known SDP with large solutions is in our normal form (SDP').**
- **Paper:** `https://arxiv.org/abs/2103.00041`

Thank you!