How to construct any weakly infeasible semidefinite program and bad projection of the psd cone?

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Given $\{A_1, \ldots, A_m\}$, define $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ by:

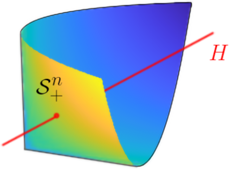
$$\mathcal{A}(X) = \begin{pmatrix} A_1 \bullet X \\ \vdots \\ A_m \bullet X \end{pmatrix}$$

Define the SDP:

$$\begin{array}{rcl} \mathcal{A}(X) & = & b \\ X & \succeq & 0 \end{array} \qquad (P)$$

With $\mathcal{A}(X_0) = b$ and $H := X_0 + \mathcal{N}(\mathcal{A})$ we have

$$\text{feas}(P) = H \cap \mathcal{S}^n_+$$



Infeasible (P) are *strongly inf.* if $\operatorname{dist}(H, \mathcal{S}_+^n) > 0$ otherwise are *weakly inf.*

Classical Example:

$$\begin{array}{rcl} x_{11} & = & 0 \\ x_{12} = x_{21} & = & 1 \\ X & \in & \mathcal{S}_+^2 \end{array} \qquad \text{(SE)}$$

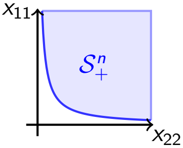
If X satisfies the equalities of (SE) then

$$X = \begin{pmatrix} 0 & 1 \\ 1 & x_{22} \end{pmatrix}$$

and so (SE) is infeasible. However, such X approach $\mathcal{S}_+^2$ by choosing

$$\begin{pmatrix} 1/x_{22} & 1 \\ 1 & x_{22} \end{pmatrix} \in \mathcal{S}_+^2$$

and making x_{22} large. We visualize the 2×2 psd matrices with $x_{12} = 1$



Main idea: we express H in two ways.

With equations $A_1 \bullet X = 0$ and $A_2 \bullet X = 2$ where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This representation certifies that (SE) is *infeasible*.

Moreover, $H = \{\lambda X_1 + X_2 \mid \lambda \in \mathbb{R}\}$ where

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

proves H is an asymptote of $\mathcal{S}_+^n$ and thus (SE) is *not-strongly infeasible*.

We see that A_1, A_2, X_1, X_2 share a common "echelon" structure.

We show every SDP can be "untangled" into such a form via row operations and congruence transforms.

Why study weak infeasibility?

Classically...

- as asymptotes of the semidefinite cone. [Klee 1961]

In modern literature...

- as hard SDPs, identified as feasible even by state-of-the-art solvers. [Liu, Pataki 2017]
- as *infeasible* and *ill-posed* SDPs with poor IPM performance. [Peña, Renegar 2000 – Bürgisser, Cucker 2013]
- as non-closed projections of $\mathcal{S}_+^n$ where $b \in \mathrm{cl}(\mathcal{A}(\mathcal{S}_+^n))$: studied as projective varieties of the Grassmanian. [Jiang, Sturmfels 2020]
- as instances in the Lasserre hierarchy of polynomial optimization. [Henrion, Lasserre 2005]
- and many more! (See references)

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Question...

What is the feasibility status of (BE)?

$$\begin{aligned}
\begin{pmatrix} -54 & -15 & 36 & -81 \\ -15 & 0 & -9 & -15 \\ 36 & -9 & 6 & 45 \\ -81 & -15 & 45 & -126 \end{pmatrix} \bullet X &= 18 \\
\begin{pmatrix} -4 & -2 & -24 & 18 \\ -2 & 0 & 2 & -2 \\ -24 & 2 & -12 & -26 \\ 18 & -2 & -26 & 44 \end{pmatrix} \bullet X &= -4 \\
\begin{pmatrix} -18 & 4 & 30 & -46 \\ 4 & 0 & -2 & 4 \\ 30 & -2 & 16 & 32 \\ -46 & 4 & 32 & -78 \end{pmatrix} \bullet X &= 4 \\
X &\succeq 0
\end{aligned} \qquad \text{(BE)}$$

Answer should provide *certificate* of such status.

Motivation

$$a_i^\top x \;=\; b_i, \quad i = 1, \ldots, m \tag{1}$$

is infeasible $\iff$ row echelon form gives

$$0^\top x = 1 \tag{2}$$

(2) *certifies* infeasibility for (1).

Append (2) to linear system to generate *any* infeasible system.

Goal: Find a similar characterization for *weakly infeasible* SDPs

To do this, we'd like analogues to

1. elementary row operations
2. row echelon form

for SDPs...

We reformulate (P) with invertible elementary operations:

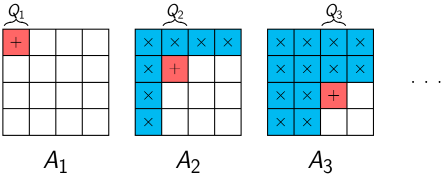
- Exchange (A_i, b_i) and (A_j, b_j)
- Add multiple of (A_i, b_i) to (A_j, b_j)
- Apply suitable $T^\top()T$ to all A_i

To extend the notion of a row echelon form, recall

a_1	1			
a_2	−2	1		
a_3	3	0	1	

$\Longrightarrow$ $\{a_1, a_2, a_3\}$ is in *row echelon form*

Similarly, $\{A_1, \dots, A_k\}$ is in *semidefinite echelon form* with *structure* $\{Q_1, \dots, Q_k\}$ if its form is



or any permutation $P^\top()P$ applied to elements of the above sequence.

Main Result: Certificates of weak infeasibility

(P) is *weakly infeasible* $\iff$ it has reformulation

$$\begin{array}{rcl} A_i' \bullet X & = & 0, \quad i = 1, \ldots, k \\ A_{k+1}' \bullet X & = & -1 \\ & \vdots & \\ X & \succeq & 0 \end{array}$$

with $\{A_1', \ldots, A_{k+1}'\}$ in *semidefinite echelon form*, and for

some $\{X_1, \ldots, X_{\ell+1}\}$ in *semidefinite echelon form* we have

$$\begin{array}{rcl} \mathcal{A}'(X_i) & = & 0, \quad i = 1, \ldots, \ell \\ \mathcal{A}'(X_{\ell+1}) & = & b'. \end{array}$$

Note, here we understand

$$\mathcal{A}'(X) = (A_1' \bullet X, \ldots, A_m' \bullet X)^\top$$

Feasibility of (BE)

Returning to (BE) and reformulating:

$\{A'_1, \cdots, A'_3\}$ certify *infeasibility* with $b' = (0, 0, -2)^\top$

$$
\underset{A'_1}{\begin{array}{r|c|c|c|c|}
\cline{2-5}
P_1\{ & 2 & & & \\ \cline{2-5}
P_2\{ & & & & \\ \cline{2-5}
P_3\{ & & & & \\ \cline{2-5}
 & & & & \\ \cline{2-5}
\end{array}}
\quad
\underset{A'_2}{\begin{array}{|c|c|c|c|}
\hline
-2 & 4 & -4 & 1 \\ \hline
4 & 2 & & \\ \hline
-4 & & & \\ \hline
1 & & & \\ \hline
\end{array}}
\quad
\underset{A'_3}{\begin{array}{|c|c|c|c|}
\hline
2 & -2 & 0 & 2 \\ \hline
-2 & 2 & -2 & 1 \\ \hline
0 & -2 & 2 & \\ \hline
2 & 1 & & \\ \hline
\end{array}}
$$

while $\{X_1, \cdots, X_3\}$ certify *not strong infeasibility*

$$
\underset{X_1}{\begin{array}{r|c|c|c|c|}
\cline{2-5}
 & & & & \\ \cline{2-5}
Q_2\{ & & & & \\ \cline{2-5}
Q_3\{ & & & & \\ \cline{2-5}
Q_1\{ & & & & 1 \\ \cline{2-5}
\end{array}}
\quad
\underset{X_2}{\begin{array}{|c|c|c|c|}
\hline
 & & & -1 \\ \hline
 & 1 & & 1 \\ \hline
 & & & -2 \\ \hline
-1 & 1 & -2 & 2 \\ \hline
\end{array}}
\quad
\underset{X_3}{\begin{array}{|c|c|c|c|}
\hline
 & 1 & & -4 \\ \hline
1 & 0 & 1 & 10 \\ \hline
 & 1 & 1 & 0 \\ \hline
-4 & 10 & 0 & 0 \\ \hline
\end{array}}
$$

Proving (BE) is *weakly infeasible*

$$
\begin{array}{c}
\begin{array}{r} P_1 \\ P_2 \\ P_3 \\ \end{array}
\begin{bmatrix} 2 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \\
A_1'
\end{array}
\qquad
\begin{array}{c}
\begin{bmatrix} -2 & 4 & -4 & 1 \\ 4 & 2 & & \\ -4 & & & \\ 1 & & & \end{bmatrix} \\
A_2'
\end{array}
\qquad
\begin{array}{c}
\begin{bmatrix} 2 & -2 & 0 & 2 \\ -2 & 2 & -2 & 1 \\ 0 & -2 & 2 & \\ 2 & 1 & & \end{bmatrix} \\
A_3'
\end{array}
$$

$\{A_1', \cdots, A_3'\}$ certifies *infeasibility* since

$$
\begin{aligned}
\text{Assume } X \text{ feasible} \quad &\Longrightarrow \quad \text{1st row of } X \text{ is } 0 \\
&\Longrightarrow \quad \text{2nd row of } X \text{ is } 0 \\
&\Longrightarrow \quad A_3' \bullet X \geq 0 \\
&\Longrightarrow \quad \textit{contradicts } A_3' \bullet X = -2
\end{aligned}
$$

Meanwhile...

To see that (BE) is *not strongly infeasible*:

Permuting columns, we write $\{X_1, \ldots, X_3\}$ as

$$
\underbrace{\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}}_{X_1}
\qquad
\underbrace{\begin{array}{|c|c|c|c|}\hline 2 & 1 & -2 & -1 \\ \hline 1 & 1 & & \\ \hline -2 & & & \\ \hline -1 & & & \\ \hline \end{array}}_{X_2}
\qquad
\underbrace{\begin{array}{|c|c|c|c|}\hline 0 & 10 & 0 & -4 \\ \hline 10 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & \\ \hline -4 & 1 & & \\ \hline \end{array}}_{X_3}
$$

Fix $\epsilon > 0$ and perturb $(4,4)$ block $\implies$ build positive definite certificate as

$$
\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \underbrace{\epsilon}_{\succ 0} \\ \hline \end{array}
\;\xmapsto{+X_3}\;
\underbrace{
\begin{array}{|c|c|c|c|}\hline 0 & 10 & 0 & -4 \\ \hline 10 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & \\ \hline -4 & 1 & & \epsilon \\ \hline \end{array}
\;\xmapsto[\gamma_2 \gg 0]{+\gamma_2 X_2}\;
\begin{array}{|c|c|c|c|}\hline \times & \times & \times & \times \\ \hline \times & + & \times & \times \\ \hline \times & \times & + & \\ \hline \times & \times & & \epsilon \\ \hline \end{array}
\;\xmapsto[\gamma_1 \ggg 0]{+\gamma_1 X_1}\;
\underbrace{\begin{array}{|c|c|c|c|}\hline + & \times & \times & \times \\ \hline \times & + & \times & \times \\ \hline \times & \times & + & \\ \hline \times & \times & & \epsilon \\ \hline \end{array}}_{\succ 0}
}_{\epsilon\text{-close to } H' := \{X \mid \mathcal{A}'X = b'\}}
$$

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Main Algorithm

By Main Result, generating any *weakly infeasible* (P) is done simply:

1. Pick $\{A_1, \ldots, A_{k+1}\}$ *SDE form* to certify *infeasibility*.
2. Pick $\{X_1, \ldots, X_{\ell+1}\}$ *SDE form* to certify *not strong infeasibility*.
3. Adjust to satisfy:

$$A_i \bullet X_j = \begin{cases} 0 & \text{if } (i,j) \neq (k+1, \ell+1) \\ -1 & \text{if } (i,j) = (k+1, \ell+1) \end{cases} \qquad (BASE)$$

4. Create other equalities $A_i \bullet X = b_i$ for $i = k+2, \ldots, m$

In step 3, we simultaneously adjust elements of A_i and $X_1, \ldots, X_{\ell+1}$.

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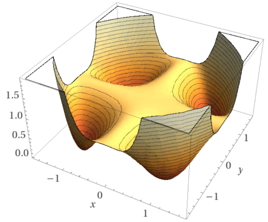
Do we always have to reformulate the SDP? ...sometimes we don't!

Consider minimizing the non-convex Motzkin polynomial

$$f(x, y) = 1 - 3x^2y^2 + x^2y^4 + x^4y^2$$



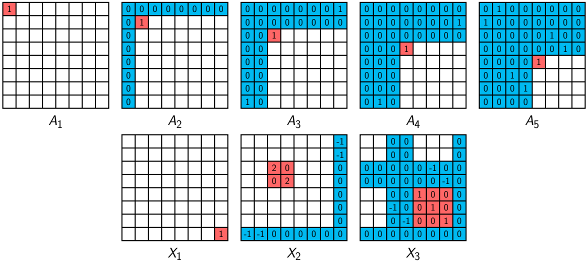
using the degree r sum-of-squares relaxation:

$$\begin{array}{rcc} \sup & \lambda & \\ f - \lambda & = & Q \bullet zz^\top \\ Q & \succeq & 0 \end{array}$$

where $z = (1, x, y, x^2, xy, y^2, \ldots, x^r, \ldots, y^r)^\top$ for $r \geq 3$.

Well known that $f(x, y)$ is non-negative but not a sum of squares.

Thus, every SOS relaxation is *infeasible*. However... for relaxation order $r \geq 3$, *infeasibility* and *not-strong infeasibility* certificates such as



are found directly in constraints: certify relaxations are *weakly infeasible*.

Upshot: solvers will find minima for polynomials ϵ-close to $f(x, y)$.

A Few Relevant Papers

- Klee 1961: Asymptotes and projections of convex sets
- Liu, Pataki 2017: Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming
- Peña, Renegar 2000: Computing approximate solutions for convex conic systems of constraints
- Lourenço, Muramatsu, Tsuchiya 2015: A structural geometrical analysis of weakly infeasible SDPs
- Henrion, Lasserre 2005: Detecting global optimality and extracting solutions in gloptipoly
- Jiang, Sturmfels 2020: Bad projections of the psd cone
- Waki 2011: How to generate weakly infeasible semidefinite programs via Lasserre's relaxations for polynomial optimization
- Drusvyatskiy, Wolkowicz 2017: The many faces of degeneracy in conic optimization
- Bürgisser, Cucker 2013: Condition
- Pataki 2013: Strong duality in conic linear programming: facial reduction and extended duals

Thank you!