How do exponential size solutions arise in semidefinite programming?

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Introduction

- Semidefinite Program (SDP) Feasibility

$$\sum_{i=1}^{m} x_i A_i + B \succeq 0 \qquad (SDP)$$

where

  - A_i, B symmetric matrices, $S \succeq 0 \implies S$ is positive semidefinite
  - Size of solution: number of bits needed to encode it

Exponential size solutions in SDP: Khachiyan example

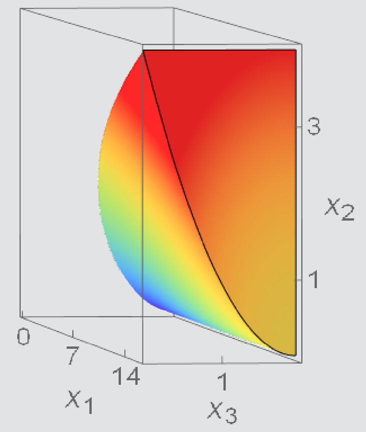
- Khachiyan example (feasibility)

$$x_1 \geq x_2^2, \quad x_2 \geq x_3^2, \quad \ldots \quad x_{m-1} \geq x_m^2, \quad x_m \geq 2 \qquad (Khachiyan)$$

- x feasible $\implies x_1 \geq 2^{2^{m-1}} \implies$ size of $x \geq \log 2^{2^{m-1}} = 2^{m-1}$
  - $m = 10 \implies x_1$ is larger than number of atoms in universe!
- Written as SDP:

$$x_i \geq x_{i+1}^2 \iff \begin{pmatrix} x_i & x_{i+1} \\ x_{i+1} & 1 \end{pmatrix} \succeq 0 \quad \forall i$$

- Feasible set (when $m = 3$)



Major open problems

1. Is SDP feasibility in P?
   - Exponential size solutions are a major obstacle
   - How to prove in polynomial time that exponential size solutions exists?
2. Can we represent large solutions in polynomial space?
   - $(Khachiyan)$ gives hope: system certifies $x_1 = 2^{2^{m-1}}$ feasible symbolically
3. Are large solutions common in SDPs? (Perhaps not...)
   - Not in "typical" SDPs in literature
   - May be eliminated in Khachiyan simply by random change of variables

     $x \leftarrow Gx$, where G is invertible matrix
   - Apparent common consent: large variables in SDPs are rare

Theorem 1: large variables are not so rare

- $\exists M$ invertible matrix, so after change of variables $x \leftarrow Mx$, if x is strictly feasible and x_k large then

$$x_1 \geq d_2 x_2^{\alpha_2}, \quad x_2 \geq d_3 x_3^{\alpha_3}, \quad \ldots \quad x_{k-1} \geq d_k x_k^{\alpha_k}$$

where

$$2 \geq \alpha_j \geq 1 + \frac{1}{k-j+1} \qquad j = 2, \ldots, k$$

and d_j, α_j are constants depending on A_i, B and fixed $x_{k+1}, \ldots, x_m$ where $k =$ singularity degree of $\{Y \succeq 0 \mid A_i \bullet Y = 0 \ \forall i\}$

- Khachiyan type hierarchy in all strictly feasible SDPs
- Assumptions we make are minimal

Examples

- Worst Case (Khachiyan SDP):

$$\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & & x_3 \\ & & x_3 & & x_4 \\ & & & x_4 & \\ x_2 & x_3 & x_4 & & 1 \end{pmatrix} \succeq 0 \implies x_1 \geq x_2^2, \quad x_2 \geq x_3^2, \quad x_3 \geq x_4^2$$

- Best Case (Mild SDP):

$$\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix} \succeq 0 \implies x_1 \geq x_2^{4/3}, \quad x_2 \geq x_3^{3/2}, \quad x_3 \geq x_4^2$$

After change of variables the SDP looks like...

$$x_1 \begin{pmatrix} \overset{r_1}{I} & \overset{n-r_1}{0} \\ 0 & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} \overset{r_1+\ldots+r_{i-1}}{\times} & \overset{r_i}{\times} & \overset{n-r_1-\ldots-r_i}{\times} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \qquad (SDP')$$

$$+ \sum_{i=k+1}^{m} x_i A'_i + B' \succeq 0$$

where $r_1, \ldots, r_k > 0$.

- A kind of "echelon form"
- Based on facial reduction
- From (SDP') we can compute the α_i exponents
- Formula to do that is akin to a continued fractions formula

SDPs naturally in the form of (SDP')

In general, we need change variables $x \leftarrow Mx$. But often, we don't. Many SDPs are naturally in the form of (SDP')!

- Example 1: Minimize $f(x) =$ univariate degree $2n$ polynomial.
  - $\rightarrow$ sum-of-squares SDP, dual looks like ($n = 3$):

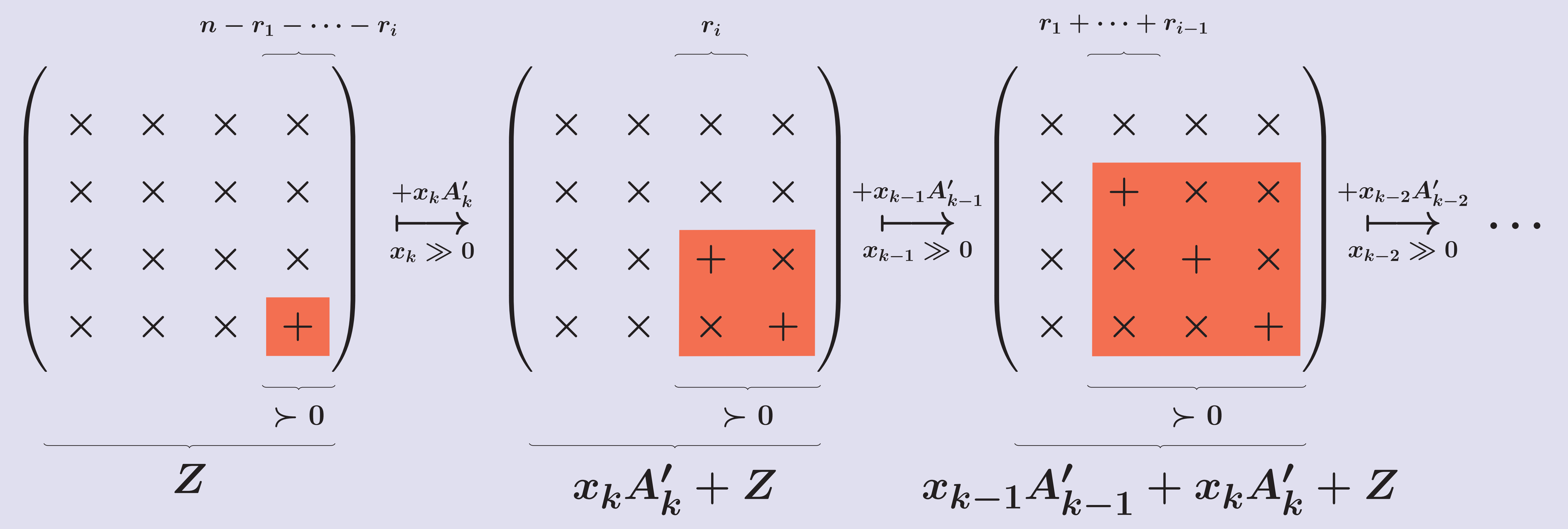
$$y_6 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + y_4 \begin{pmatrix} 0 & & 1 & \\ & 1 & & \\ 1 & & 0 & \\ & & & 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} + \cdots \succeq 0$$

Already in form of (SDP')! $\Rightarrow$ in a feasible solution $y_{2n} \geq y_2^n$

- Example 2: O'Donnell, 2017 certify non-negativity of polynomial over simple set
  - Resulting SDP is equivalent to $(Khachiyan)$, in the form of (SDP')!
- All known SDPs with exponential sized solutions are in the form of (SDP')!

How to certify exponential size solutions in polynomial space?

In (SDP') suppose $x_{k+1}, \ldots, x_m$ are part of strictly feasible solution. Can compute $x_k, \ldots, x_1$. Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$



Grow the lower right corner into a positive definite matrix

No need to actually write down $x_k, \ldots, x_1$: argument proves they exist!

Conclusions

- Khachiyan type hierarchy among leading variables in every strictly feasible SDP (after linear change of variables)
- Partial answer to: how to represent exponential size solutions in polynomial space?
- Every known SDP with large solutions is in our normal form
- Paper: `https://arxiv.org/abs/2103.00041`

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