How do Exponential Size Solutions Arise in Semidefinite Programming?

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Joint work with Alex Touzov
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Linear Programming (LP) feasibility

\[ \exists \? x \text{ s.t.} \]

\[ Ax \geq b \quad (LP) \]

Here

- \( A \in \mathbb{Z}^{m \times n} \), \( b \in \mathbb{Z}^{m} \)
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- \( A \in \mathbb{Z}^{m \times n}, \; b \in \mathbb{Z}^m \)
- Poly size solutions: if (LP) feasible \( \Rightarrow \exists \) feasible rational \( \bar{x} \) in which entries have numerator and denominator with size \( \leq n \log n \log L \)

where \( L = \) largest entry in \( A, b. \)
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  where \( L = \) largest entry in \( A, b \).
  Use Kramer’s rule at an extreme point of (LP).
- \( \rightarrow \) To solve (LP) in poly time, we find a solution \( \bar{x} \).
Semidefinite Programing (SDP) feasibility

\[ \exists \? x \text{ s.t. } \sum_{i=1}^{m} x_i A_i + B \succeq 0 \]  \hfill \text{(SDP)}

Here

- \( A_i, B \) are symmetric matrices,
- \( S \succeq 0 \) means that \( S \) is symmetric positive semidefinite (psd).
- Far reaching generalization of LP.
In SDPs exponential size solutions are unavoidable

- Khachiyan example

\[ x_1 \geq x_2^2, \ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2, \ x_m \geq 2. \quad \text{(Khachiyan)} \]
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• Khachiyian example

\[ x_1 \geq x_2^2, \ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2, \ x_m \geq 2. \] (Khachiyian)

• \( x \) feasible \( \Rightarrow x_1 \geq 2^{2^{m-1}}. \)
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(Khachyian)

• \( x \) feasible \( \Rightarrow \) \( x_1 \geq 2^{2^{m-1}} \).

• Size of \( x \) \( \geq \log 2^{2^{m-1}} = 2^{m-1} \).
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- \( x \) feasible \( \Rightarrow \ x_1 \geq 2^{2^{m-1}} \).

- Size of \( x \geq \log 2^{2^{m-1}} = 2^{m-1} \).

- Can be written as SDP:
  \[ x_i \geq x_{i+1}^2 \iff \begin{pmatrix} x_i & x_{i+1} \\ x_{i+1} & 1 \end{pmatrix} \succeq 0 \ \forall i. \]
$x_1 \geq x_2^2$, $x_2 \geq x_3^2$, $2 \geq x_3 \geq 0$ (1)
Is (SDP) feasibility in P?

- Major open problem
- Open even for quadratic constraints
Is \((\text{SDP})\) feasibility in \(P\)?

- Major open problem
- Open even for quadratic constraints
- Exponential size solutions are a major obstacle
- How to prove in polynomial time that a possibly exponential size solution exists?
Question 1

• Can we represent such large solutions in polynomial space?
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• (Khachiyan) gives hope: no need to write out $2^{2^{m-1}}$ to convince ourselves that $x_1 = 2^{2^{m-1}}$ is feasible.
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- (Khachiyan) gives hope: no need to write out $2^{2^{m-1}}$ to convince ourselves that $x_1 = 2^{2^{m-1}}$ is feasible.
- The system itself is a certificate.
Question 2
Are large solutions common in SDPs?

Seemingly *no*, since:
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  (1) replace

  \[ x_m \geq 2 \rightarrow x_m \geq 2 + x_{m+1} \]

  where \( x_{m+1} \) is a new variable

  \( \rightarrow x_1 \) does not have to be large anymore.
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\begin{itemize}
\item they do not come up in LPs, or \textquotedblleft typical\textquotedblright SDPs.
\item we may eliminate them even in (Khachiyan) by a very slight change, as:
\end{itemize}

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(2) by linear change of variables:

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x \leftarrow Gx
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where $G$ is random dense matrix.

$\rightarrow$ (Khachiyan) becomes a big mess.
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  2. by linear change of variables:
     
     \[ x \leftarrow Gx \]
     
     where \( G \) is random dense matrix.
     
     \( \rightarrow \) (Khachiyan) becomes a big mess.

\( \rightarrow \) Apparent common consent: large variables in SDPs are rare.
However: Main result (informal)

- We can “untangle” any strictly feasible SDP and make it into a Khachiyan type SDP.
Background

\( k := \) singularity degree of \( \{ Y \geq 0 : A_i \bullet Y = 0 \forall i \} \).
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- $k := \text{singularity degree of } \{ Y \succeq 0 : A_i \bullet Y = 0 \forall i \}$.

- minimum number of facial reduction steps to certify maximum rank psd matrix
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- $k \leq 1$ when (SDP) is an LP.
Background

• $k :=$ singularity degree of $\{ Y \succeq 0 : A_i \bullet Y = 0 \forall i \}$.

• minimum number of facial reduction steps to certify maximum rank psd matrix

• $k \leq 1$ when (SDP) is an LP.

• We assume that (SDP) is strictly feasible, i.e., $\exists x$ s.t.

\[
\sum_{i=1}^{m} \ x_i A_i + B \succ 0.
\]
Theorem 1 (Informal)

After a linear change of variables $x \leftarrow Mx$, if $x$ strictly feasible and $x_k$ is large, then

$$x_1 \geq d_2 x_2^{\alpha_2}, \quad x_2 \geq d_3 x_3^{\alpha_3}, \ldots, \quad x_{k-1} \geq d_k x_k^{\alpha_k}$$
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where

$$2 \geq \alpha_2 \geq \frac{k}{k-1}, \quad 2 \geq \alpha_3 \geq \frac{k-1}{k-2}, \ldots, \quad 2 \geq \alpha_k \geq 2.$$
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The $d_j$ and $\alpha_j$ are constants that depend on the $A_i$, on $B$ and $x_{k+1}, \ldots, x_m$ that we consider fixed.

Khachiyan type hierarchy in all strictly feasible SDPs.
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Khachiyan type hierarchy in all strictly feasible SDPs.

Assumptions are minimal.
Corollary

• In worst case (all $\alpha_j = 2$)

$$x_1 \geq \text{constant} \cdot x_k^{2^{k-1}}.$$
Corollary

• In worst case (all $\alpha_j = 2$)
  $$x_1 \geq \text{constant} \cdot x_k^{2k-1}.$$  

• In best case (all $\alpha_j = \text{lower bound}$)
  $$x_1 \geq \text{constant} \cdot x_k^k.$$
Worst case example: Khachiyan SDP

\[
\begin{pmatrix}
  x_1 & x_2 \\
  x_2 & x_3 \\
  x_3 & x_4 \\
  x_4 & 1
\end{pmatrix} \succeq 0
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\]

- Subdeterminant with three red corners $\Rightarrow x_1 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$
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Exponents are maximal.
Best case example: “Mild” SDP

\[
\begin{pmatrix}
x_1 & x_2 \\
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x_2 & x_3 & x_4 \\
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x_4 & 1
\end{pmatrix} \succeq 0
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Best case example: “Mild” SDP

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\begin{pmatrix}
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  x_2 & x_3 \\
  x_2 & x_3 & x_4 \\
  x_3 & x_4 \\
  x_4 & 1
\end{pmatrix} \succeq 0
\]

- Subdeterminant with three red corners $\Rightarrow x_1 x_3 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 x_4 \geq x_3^2$
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\end{pmatrix} \succeq 0
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- Subdeterminant with three red corners \( \Rightarrow x_1 x_3 \geq x_2^2 \)
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From these we derive:

\[ x_1 \geq x_2^{4/3}, \quad x_2 \geq x_3^{3/2}, \quad x_3 \geq x_4^2 \]

Exponents are minimal.
Khachiyann vs Mild

- Three variables, $2 \geq x_3 \geq 0$ (normalization)
Proof idea

(1) \((SDP) \rightarrow \text{reformulate (change variables) to get (SDP')}\)
Proof idea

(1) (SDP) $\rightarrow$ reformulate (change variables) to get (SDP’)

(2) (SDP’) $\rightarrow$ messy quadratic inequalities such as

$$(x_1 + 2x_2 + 5x_3)(x_4 + x_5) > (x_2 - 3x_6)^2$$
Proof idea

(1) (SDP) $\rightarrow$ reformulate (change variables) to get (SDP')

(2) (SDP') $\rightarrow$ messy quadratic inequalities such as

$$(x_1 + 2x_2 + 5x_3)(x_4 + x_5) > (x_2 - 3x_6)^2$$

(3) messy quadratic inequalities $\rightarrow$ cleaned up inequalities such as

$$x_1x_4 > \text{constant} \cdot x_2^2 \text{ if } x_k \text{ large}$$
Proof idea

(1) \((\text{SDP}) \longrightarrow \text{reformulate (change variables) to get } (\text{SDP'})\)

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(x_1 + 2x_2 + 5x_3)(x_4 + x_5) > (x_2 - 3x_6)^2
\]

(3) \(\text{messy quadratic inequalities} \longrightarrow \text{cleaned up inequalities such as}\)

\[
x_1x_4 > \text{constant} \cdot x_2^2 \text{ if } x_k \text{ large}
\]

+ eliminate variables to get

\[
x_j \geq \text{constant} \cdot x_{j+1}^{\alpha_{j+1}} \forall j
\]

+ recursion to compute the \(\alpha_{j+1}\).
Reformulating (SDP) into (SDP’)

The reformulated SDP looks like

\[ x_1 \begin{pmatrix} r_1 & n-r_1 \\ I & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i A_i' + B' \succeq 0 \]

(SDP’)

with \( r_1, \ldots, r_k > 0 \).
Reformulating (SDP) into (SDP')

The reformulated SDP looks like

\[
x_1 \begin{pmatrix} r_1 & n-r_1 \\ I & 0 & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} r_1+\ldots+r_{i-1} & r_i & n-r_1-\ldots-r_i \\ \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}
\]

\[+ \sum_{i=k+1}^{m} x_i A'_i + B' \succeq 0 \quad \text{(SDP')}
\]

with \( r_1, \ldots, r_k > 0 \).

To get this reformulation, we also used similarity transformations \( T^\top (\cdot) T \).
(SDP’) $\rightarrow$ messy quadratic inequalities

\[ \cdots + x_2 \left( \begin{array}{cccccc} x & x & x & x & x & x \\ x & 1 & & & & \\ x & & & & & \\ x & & & & & \\ x & & & & & \end{array} \right) + x_3 \left( \begin{array}{cccc} x & x & x & x & x \\ x & 2 & 4 & & x \\ x & & 1 & & \\ x & & 4 & & \\ x & & & & \end{array} \right) + x_4 \left( \begin{array}{cccc} x & x & x & x & x \\ x & 5 & 7 & & x \\ x & & 7 & & x \\ x & & & 1 & \end{array} \right) + \cdots > 0 \]
(SDP’) $\rightarrow$ messy quadratic inequalities

\[
\cdots + x_2 \begin{pmatrix} x & x & x & x & x & x \\ x & 1 \end{pmatrix} + x_3 \begin{pmatrix} x & x & x & x & x & x \\ x & 1 \end{pmatrix} + x_4 \begin{pmatrix} x & x & x & x & x & x \\ x & 1 \end{pmatrix} + \cdots > 0
\]

- $2 \times 2$ subdeterminant $\rightarrow$

\[
(x_2 + 2x_3 + 5x_4 + \ldots)(x_4 + \ldots) > (4x_3 + 7x_4 + \ldots)^2
\]
(SDP') → messy quadratic inequalities

\[ \cdots + x_2 \left( \begin{array}{ccccccc} \times & \times & \times & \times & \times & \times \\ \times & 1 \\ \times \\ \times \\ \times \end{array} \right) + x_3 \left( \begin{array}{ccccccc} \times & \times & \times & \times & \times & \times \\ \times & 2 & \times & 4 & \times & \times \\ \times & \times & 1 \\ \times & \times \\ \times \end{array} \right) + x_4 \left( \begin{array}{ccccccc} \times & \times & \times & \times & \times & \times \\ \times & 5 & \times & 7 & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{array} \right) + \cdots \geq 0 \]

• 2 × 2 subdeterminant →

\[(x_2 + 2x_3 + 5x_4 + \ldots)(x_4 + \ldots) \geq (4x_3 + 7x_4 + \ldots)^2\]

• The \ldots mean a combination of higher numbered variables.
\[(SDP') \rightarrow \text{messy quadratic inequalities}\]

\[
\begin{align*}
\cdots + x_2 \left( \begin{array}{ccccccc}
\times & \times & \times & \times & \times & \times & \times \\
\times & 1 \\
\times \\
\times \\
\times \\
\times
\end{array} \right) & + x_3 \left( \begin{array}{ccccccc}
\times & \times & \times & \times & \times & \times & \times \\
\times & 2 & \times & 4 & \times & \times \\
\times & 4 \\
\times & \times \\
\times & \times \\
\times
\end{array} \right) & + x_4 \left( \begin{array}{ccccccc}
\times & \times & \times & \times & \times & \times & \times \\
\times & 5 & \times & 7 & \times & \times \\
\times & 7 & \times & 1 \\
\times & \times \\
\times & \times \\
\times
\end{array} \right) & + \cdots > 0
\end{align*}
\]

- **2 \times 2** subdeterminant →

\[
(x_2 + 2x_3 + 5x_4 + \ldots)(x_4 + \ldots) > (4x_3 + 7x_4 + \ldots)^2
\]

- The \ldots mean a combination of higher numbered variables.
- Cleaned up version (suppress higher numbered terms):

\[
x_2x_4 > \text{const} \cdot x_3^2, \quad \text{if} \ x_k \text{ is large}
\]
How to compute the $\alpha_{j+1}$ in $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$?

$$\alpha_{j+1} = \begin{cases} 
2 - \frac{1}{\alpha_{j+2} \ldots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\
2 & \text{if } t_{j+1} = k + 1
\end{cases}$$

for $j = 1, \ldots, k - 1$. 
How to compute the $\alpha_{j+1}$ in $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$?

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2 - \frac{1}{\alpha_{j+2} \ldots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\
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Similar to continued fractions.
How to compute the $\alpha_{j+1}$ in $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$?

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Similar to continued fractions.

Here

$$t_{j+1} = \text{index of a rightmost block with } x_{j+1}$$

Shift $x_{j+1}$ to right $\Rightarrow t_{j+1}$ increases.
How to compute the $\alpha_{j+1}$ in $x_j \geq \text{const} \cdot x_j^{\alpha_{j+1}}$?

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Here

$$t_{j+1} = \text{index of a rightmost block with } x_{j+1}$$

Shift $x_{j+1}$ to right $\Rightarrow t_{j+1}$ increases.

$\Rightarrow \alpha_{j+1}$ increases.
Example

\[ \begin{pmatrix}
  x_1 & x_2 \\
  x_2 & x_3 \\
  x_2 & x_3 & x_4 \\
  x_3 & x_4 \\
  x_4 & 1
\end{pmatrix} \]

\[ \alpha = \left( \frac{4}{3}, \frac{3}{2}, 2 \right) \]
Example

\[
\begin{pmatrix}
  x_1 & x_2 \\
  x_2 & x_3 \\
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\end{pmatrix}
\]

\[\alpha = \left(\frac{4}{3}, \frac{3}{2}, 2\right)\]

\[
\begin{pmatrix}
  x_1 & x_2 \\
  x_2 & x_3 \\
  x_2 & x_3 & x_4 \\
  x_2 & x_3 \\
  x_4 & 1
\end{pmatrix}
\]

\[\alpha = \left(\frac{5}{3}, \frac{3}{2}, 2\right)\]
Example

$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 \end{pmatrix} \xrightarrow{\alpha=(4/3, 3/2, 2)} \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 \end{pmatrix} \xrightarrow{\alpha=(5/3, 3/2, 2)} \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_2 & x_4 \end{pmatrix}$$

$$\alpha=(4/3, 3/2, 2) \quad \alpha=(5/3, 3/2, 2) \quad \alpha=(2, 3/2, 2)$$
Example

In other words

\[ x_1 \geq x_2^{4/3} \rightarrow x_1 \geq x_2^{5/3} \rightarrow x_1 \geq x_2^2 \]
Connection to Fourier-Motzkin elimination

- We are eliminating variables to get to $x_j \geq \text{const } x_{j+1}^{\alpha_j+1}$.
- This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$. 
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• Example

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\begin{align*}
    x_1x_3 & \geq x_2^2 \\
    x_2x_4 & \geq x_3^2 \\
    x_3 & \geq x_4^2
\end{align*}
\]
Connection to Fourier-Motzkin elimination

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- Example

\[
\begin{align*}
x_1x_3 & \geq x_2^2 & y_1 + y_3 & \geq 2y_2 \\
x_2x_4 & \geq x_3^2 \quad \rightarrow \quad y_2 + y_4 & \geq 2y_3 \\
x_3 & \geq x_4^2 & y_3 & \geq 2y_4
\end{align*}
\]
Connection to Fourier-Motzkin elimination

- We are eliminating variables to get to $x_j \geq x_j^{\alpha j+1}$.
- This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$.
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\begin{align*}
    x_1 x_3 & \geq x_2^2 & \quad y_1 + y_3 & \geq 2y_2 \\
    x_2 x_4 & \geq x_3^2 & \quad y_2 + y_4 & \geq 2y_3 \\
    x_3 & \geq x_4^2 & \quad y_3 & \geq 2y_4
\end{align*}
\]

- Add $1/2$ times the last to the middle:

\[
y_2 \geq \frac{3}{2}y_3
\]
Connection to Fourier-Motzkin elimination

• We are eliminating variables to get to $x_j \geq x_{j+1}^{\alpha_j+1}$.

• This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$.

• Example

\[
\begin{align*}
x_1 x_3 & \geq x_2^2 & \quad & y_1 + y_3 \geq 2y_2 \\
x_2 x_4 & \geq x_3^2 & \quad & \rightarrow \quad y_2 + y_4 \geq 2y_3 \\
x_3 & \geq x_4^2 & \quad & y_3 \geq 2y_4
\end{align*}
\]

• Add $1/2$ times the last to the middle:

\[
x_2 \geq x_3^{3/2} \quad \leftarrow \quad y_2 \geq \frac{3}{2} y_3
\]
Do we need the change of variables \( x \leftarrow Mx \)?

- In general, yes: such an operation may mess up even (Khachiyan).
- So, we may need such an operation \( x \leftarrow M^{-1}x \) to unmess it.
- But, sometimes we don’t.
When we do not even need a change of variables, part 1

Want to minimize $f(x) = \text{univariate degree } 2n \text{ polynomial}.$

Rewrite as SDP, using sum-of-squares technique, look at dual (show the case $n = 3$)

$$y_6 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y_4 \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + \cdots \succeq 0$$
When we do not even need a change of variables, part 1

Want to minimize $f(x) = \text{univariate degree } 2n \text{ polynomial}$. Rewrite as SDP, using sum-of-squares technique, look at dual (show the case $n = 3$)

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Exactly in the form of (SDP'), without a change of variables.
When we do not even need a change of variables, part 1

Want to minimize $f(x) = \text{univariate degree } 2n \text{ polynomial}$. Rewrite as SDP, using sum-of-squares technique, look at dual (show the case $n = 3$)

\[
y_6 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y_4 \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \cdots \geq 0
\]

Exactly in the form of (SDP$'$), without a change of variables.

**Corollary:** $y \in \mathbb{R}^{2n}$ feasible $\implies$

\[
y_{2n} \geq y_{2n-2}^{1+1/(n-1)}, \ y_{2n-2} \geq y_{2n-4}^{1+1/(n-2)}, \cdots
\]

\[
y_{2n} \geq y_2^n.
\]
When we do not even need a change of variables, part 2

O’ Donnell, 2017 We want to certify that a polynomial

\[ p(x, y) = x_1 + \cdots + x_n - 2y_1 \geq 0 \]

for all \((x, y) \in K\), where \(K\) is a simple set.
When we do not even need a change of variables, part 2

O’ Donnell, 2017 We want to certify that a polynomial
\[ p(x, y) = x_1 + \cdots + x_n - 2y_1 \geq 0 \]
for all \((x, y) \in K\), where \(K\) is a simple set.

Resulting SDP:
\[ u_1 E_{11} + \sum_{i=2}^{n} u_i (E_{ii} - E_{i-1,n+i-1}) + B \succeq 0. \]
Here \(E_{ij}\) is the \((i, j)\) unit matrix.

Exactly in the form of \((SDP')\)! It yields essentially Khachiyan’s example.
Certifying exponential size solutions in polynomial space, without computing them

Revisiting the reformulated problem:

\[
\begin{align*}
& x_1 \begin{pmatrix} r_1 & n-r_1 \\ I & 0 & 0 \end{pmatrix} + \sum_{i=2}^k x_i \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \\
& + \sum_{i=k+1}^m x_i A_i + B' \succeq 0
\end{align*}
\]

(SDP')
Certifying exponential size solutions in polynomial space, without computing them

Revisiting the reformulated problem:

\[ x_1 \begin{pmatrix} r_1 & n-r_1 \\ I & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} r_1+\ldots+r_{i-1} & r_i & n-r_1-\ldots-r_i \\ \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i A'_i + B' \succeq 0 \quad \text{(SDP')} \]

Suppose we have \( x_{k+1}, \ldots, x_m \) s.t. \( \exists x_1, \ldots, x_k \) so this problem is strictly feasible.
Certifying exponential size solutions in polynomial space, without computing them

Revisiting the reformulated problem:

\[ x_1 \begin{pmatrix} r_1 & n-r_1 \\ I & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} r_1 + \ldots + r_{i-1} & r_i & n-r_1-\ldots-r_i \\ \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i A'_i + B' \succeq 0 \]  

(SDP')

Suppose we have \( x_{k+1}, \ldots, x_m \) s.t. \( \exists x_1, \ldots, x_k \) so this problem is strictly feasible.

Then we can prove that \( x_1, \ldots, x_k \) exist without having to compute them.
Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^{m} x_i A'_i + B'$ positive definite.

Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$
Verifying that \(x_1, \ldots, x_k\) exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of \(\sum_{i=1}^{m} x_i A'_i + B'\) positive definite. Start with \(Z := \sum_{i=k+1}^{m} x_i A'_i + B'\).
Verifying that $x_1, \ldots, x_k$ exist, without computing them

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\[ Z := \sum_{i=k+1}^{m} x_iA'_i + B' \]
Verifying that \( x_1, \ldots, x_k \) exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of \( \sum_{i=1}^{m} x_i A'_i + B' \) positive definite. Start with \( Z := \sum_{i=k+1}^{m} x_i A'_i + B' \).

\[
\begin{pmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{pmatrix}
\overset{I_{k+1}}{\Rightarrow} + x_k A'_k \\
\begin{pmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{pmatrix}
\overset{I_k}{\Rightarrow} + x_k A'_k + Z \\
\begin{pmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{pmatrix}
\overset{I_{k-1}}{\Rightarrow} + x_{k-1} A'_{k-1} + x_k A'_k + Z
\]
Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^{m} x_i A_i' + B'$ positive definite. Start with $Z := \sum_{i=k+1}^{m} x_i A_i' + B'$.

\[
\begin{array}{c}
\begin{pmatrix}
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \end{pmatrix}
\end{array}
\begin{array}{c}
  + x_k A_k' \\
  x_k \gg 0
\end{array}
\begin{array}{c}
  \begin{pmatrix}
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \times & \times & + & + \\
  \end{pmatrix}
\end{array}
\begin{array}{c}
  + x_{k-1} A_{k-1}' \\
  x_{k-1} \gg 0
\end{array}
\begin{array}{c}
  \begin{pmatrix}
  \times & \times & \times & \times \\
  \times & \times & + & + \\
  \times & \times & + & + \\
  \times & \times & + & + \\
  \end{pmatrix}
\end{array}
\begin{array}{c}
  + x_{k-2} A_{k-2}' \\
  x_{k-2} \gg 0
\end{array}
\begin{array}{c}
  Z
\end{array}
\begin{array}{c}
  x_k A_k' + Z
\end{array}
\begin{array}{c}
  x_{k-1} A_{k-1}' + x_k A_k' + Z
\end{array}
\begin{array}{c}
  I_{k+1}
\end{array}
\begin{array}{c}
  I_k
\end{array}
\begin{array}{c}
  I_{k-1}
\end{array}
\]
Verifying that \(x_1, \ldots, x_k\) exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of \(\sum_{i=1}^{m} x_i A'_i + B'\) positive definite.

Start with \(Z := \sum_{i=k+1}^{m} x_i A'_i + B'\)

\[
\begin{pmatrix}
 x & x & x & x & x \\
 x & x & x & x & x \\
 x & x & x & x & x \\
 x & x & x & x & x \\
 x & x & x & x & x \\
 \end{pmatrix}
\begin{array}{c}
 I_{k+1} \\
 \succ 0 \\
 \end{array}

\begin{pmatrix}
 x & x & x & x & x \\
 x & x & x & x & x \\
 x & + & x & x & x \\
 x & x & + & x & x \\
 x & x & x & x & x \\
 \end{pmatrix}
\begin{array}{c}
 I_k \\
 \succ 0 \\
 \end{array}

\begin{pmatrix}
 x & x & x & x & x \\
 x & + & x & x & x \\
 x & + & x & x & x \\
 x & + & x & x & x \\
 x & x & x & x & x \\
 \end{pmatrix}
\begin{array}{c}
 I_{k-1} \\
 \succ 0 \\
 \end{array}

\begin{pmatrix}
 x & x & x & x & x \\
 x & x & + & x & x \\
 x & x & + & x & x \\
 x & x & + & x & x \\
 x & x & x & x & x \\
 \end{pmatrix}
\begin{array}{c}
 \ldots \\
 \end{array}

x_k A'_k + Z

x_{k-1} A'_{k-1} + x_k A'_k + Z

Question: Are all SDPs with large solutions in this regularized form (maybe after a similarity transformation)?
Conclusion

- Exponential size solutions in SDP, going back to famous Khachiyan example.
- Khachiyan type hierarchy among leading variables in every strictly feasible SDP (after linear change of variables)
- Formulas to compute the exponents (like continued fractions)
Conclusion

- Exponential size solutions in SDP, going back to famous Khachiyan example.
- Khachiyan type hierarchy among leading variables in every strictly feasible SDP (after linear change of variables)
- Formulas to compute the exponents (like continued fractions)
- Partial answer to: how to represent exponential size solutions in polynomial space?
- Every known SDP with large solutions is in our normal form (SDP’).

Thank you!