How do Exponential Size Solutions Arise in Semidefinite Programming?

Gábor Pataki

Department of Statistics and Operations Research

UNC Chapel Hill

Joint work with Alex Touzov

Fields Institute, May 11, 2021

Linear Programming (LP) feasibility

$\exists ?\ x$ s.t.

$$Ax \geq b \qquad (LP)$$

Here

- $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^{m}$

Linear Programming (LP) feasibility

$\exists ?\ x$ s.t.

$$Ax \geq b \qquad (LP)$$

Here

- $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$
- Poly size solutions: if (LP) feasible $\Rightarrow \exists$ feasible rational $\bar{x}$ in which entries have numerator and denominator with size

  $$\leq n \log n \log L$$

  where $L =$ largest entry in A, b.

Linear Programming (LP) feasibility

$\exists ?\, x$ s.t.

$$Ax \geq b \qquad (LP)$$

Here

- $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$
- Poly size solutions: if (LP) feasible $\Rightarrow \exists$ feasible rational $\bar{x}$ in which entries have numerator and denominator with size

  $$\leq n \log n \log L$$

  where $L =$ largest entry in A, b.

  Use Kramer's rule at an extreme point of (LP).

Linear Programming (LP) feasibility

$\exists ?\, x$ s.t.

$$Ax \geq b \qquad (LP)$$

Here

- $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$
- Poly size solutions: if (LP) feasible $\Rightarrow \exists$ feasible rational $\bar{x}$ in which entries have numerator and denominator with size

  $$\leq n \log n \log L$$

  where $L =$ largest entry in A, b.

  Use Kramer's rule at an extreme point of (LP).
- $\rightarrow$ To solve (LP) in poly time, we find a solution $\bar{x}$.

Semidefinite Programing (SDP) feasibility

$\exists ? x$ s.t.

$$\sum_{i=1}^{m} x_i A_i + B \succeq 0 \qquad \text{(SDP)}$$

Here

- A_i, B are symmetric matrices,
- $S \succeq 0$ means that S is symmetric positive semidefinite (psd).
- Far reaching generalization of LP.

In SDPs exponential size solutions are unavoidable

- **Khachiyan example**

$$x_1 \geq x_2^2,\ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2,\ x_m \geq 2. \qquad (\textbf{Khachiyan})$$

In SDPs exponential size solutions are unavoidable

- **Khachiyan example**

$$x_1 \geq x_2^2,\ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2,\ x_m \geq 2. \qquad (\textbf{Khachiyan})$$

- x feasible $\Rightarrow x_1 \geq 2^{2^{m-1}}$.

In SDPs exponential size solutions are unavoidable

- Khachiyan example

$$x_1 \geq x_2^2,\ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2,\ x_m \geq 2. \qquad (\text{Khachiyan})$$

- x feasible $\Rightarrow x_1 \geq 2^{2^{m-1}}$.
- Size of $x \geq \log 2^{2^{m-1}} = 2^{m-1}$.

In SDPs exponential size solutions are unavoidable

- **Khachiyan example**

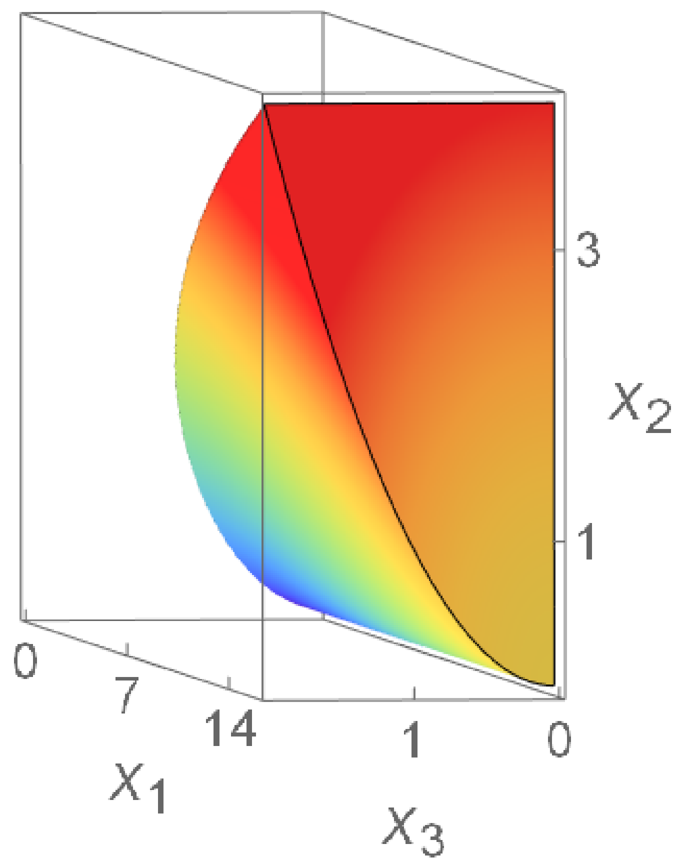
$$x_1 \geq x_2^2,\ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2,\ x_m \geq 2. \qquad (\textbf{Khachiyan})$$

- x feasible $\Rightarrow x_1 \geq 2^{2^{m-1}}$.
- Size of $x \geq \log 2^{2^{m-1}} = 2^{m-1}$.
- Can be written as SDP:

$$x_i \geq x_{i+1}^2 \Leftrightarrow \begin{pmatrix} x_i & x_{i+1} \\ x_{i+1} & 1 \end{pmatrix} \succeq 0 \ \forall i.$$

Khachiyan picture

$$x_1 \geq x_2^2,\ x_2 \geq x_3^2,\ 2 \geq x_3 \geq 0 \qquad (1)$$



Is (SDP) feasibility in P?

- **Major open problem**
- **Open even for quadratic constraints**

Is (SDP) feasibility in P?

- **Major open problem**
- **Open even for quadratic constraints**
- **Exponential size solutions are a major obstacle**
- **How to prove in polynomial time that a possibly exponential size solution exists?**

Question 1

- **Can we represent such large solutions in polynomial space?**

Question 1

- **Can we represent such large solutions in polynomial space?**
- (**Khachiyan**) **gives hope: no need to write out $2^{2^{m-1}}$ to convince ourselves that $x_1 = 2^{2^{m-1}}$ is feasible.**

Question 1

- **Can we represent such large solutions in polynomial space?**
- (**Khachiyan**) **gives hope: no need to write out $2^{2^{m-1}}$ to convince ourselves that $x_1 = 2^{2^{m-1}}$ is feasible.**
- **The system itself is a certificate.**

Question 2
Are large solutions common in SDPs?

Seemingly *no*, since:

Question 2
Are large solutions common in SDPs?

Seemingly *no*, since:

- **they do not come up in LPs, or "typical" SDPs.**

Question 2
Are large solutions common in SDPs?

Seemingly *no*, since:

- **they do not come up in LPs, or "typical" SDPs.**
- **we may eliminate them even in (Khachiyan) by a very slight change, as:**

Question 2
Are large solutions common in SDPs?

Seemingly *no*, since:

- **they do not come up in LPs, or "typical" SDPs.**
- **we may eliminate them even in (Khachiyan) by a very slight change, as:**

(1) replace

$$x_m \geq 2 \rightarrow x_m \geq 2 + x_{m+1}$$

where x_{m+1} is a new variable

$\rightarrow$ x_1 does not have to be large anymore.

Question 2
Are large solutions common in SDPs?

Seemingly *no*, since:

- **they do not come up in LPs, or "typical" SDPs.**
- **we may eliminate them even in (Khachiyan) by a very slight change, as:**

(1) replace

$$x_m \geq 2 \rightarrow x_m \geq 2 + x_{m+1}$$

where x_{m+1} is a new variable

$\rightarrow$ x_1 does not have to be large anymore.

(2) by linear change of variables:

$$x \leftarrow Gx$$

where G is random dense matrix.

$\rightarrow$ (Khachiyan) becomes a big mess.

Question 2
Are large solutions common in SDPs?

Seemingly *no*, since:

- they do not come up in LPs, or "typical" SDPs.
- we may eliminate them even in (Khachiyan) by a very slight change, as:

(1) replace

$$x_m \geq 2 \;\rightarrow\; x_m \geq 2 + x_{m+1}$$

where x_{m+1} is a new variable

$\rightarrow$ x_1 does not have to be large anymore.

(2) by linear change of variables:

$$x \leftarrow Gx$$

where G is random dense matrix.

$\rightarrow$ (Khachiyan) becomes a big mess.

$\rightarrow$ Apparent common consent: large variables in SDPs are rare.

However: Main result (informal)

- **We can "untangle" any strictly feasible SDP and make it into a Khachiyan type SDP.**

Background

- $k :=$ singularity degree of $\{\, Y \succeq 0 : A_i \bullet Y = 0 \,\forall i \,\}$.

Background

- $k :=$ singularity degree of $\{\, Y \succeq 0 : A_i \bullet Y = 0 \,\forall i \,\}$.
- minimum number of facial reduction steps to certify maximum rank psd matrix

Background

- $k :=$ singularity degree of $\{ Y \succeq 0 : A_i \bullet Y = 0 \, \forall i \}$.
- minimum number of facial reduction steps to certify maximum rank psd matrix
- $k \leq 1$ when (SDP) is an LP.

Background

- $k :=$ singularity degree of $\{ Y \succeq 0 : A_i \bullet Y = 0 \, \forall i \}$.
- minimum number of facial reduction steps to certify maximum rank psd matrix
- $k \leq 1$ when (SDP) is an LP.
- We assume that (SDP) is strictly feasible, i.e., $\exists x$ s.t.

$$\sum_{i=1}^{m} x_i A_i + B \succ 0.$$

Theorem 1 (Informal)

After a linear change of variables $x \leftarrow Mx$, if x strictly feasible and x_k is large, then

$$x_1 \geq d_2 x_2^{\alpha_2},\ x_2 \geq d_3 x_3^{\alpha_3}, \ldots,\ x_{k-1} \geq d_k x_k^{\alpha_k}$$

Theorem 1 (Informal)

After a linear change of variables $x \leftarrow Mx$, if x strictly feasible and x_k is large, then

$$x_1 \geq d_2 x_2^{\alpha_2},\ x_2 \geq d_3 x_3^{\alpha_3}, \ldots,\ x_{k-1} \geq d_k x_k^{\alpha_k}$$

where

$$2 \geq \alpha_2 \geq \frac{k}{k-1}, 2 \geq \alpha_3 \geq \frac{k-1}{k-2}, \ldots,\ 2 \geq \alpha_k \geq 2.$$

Theorem 1 (Informal)

After a linear change of variables $x \leftarrow Mx$, if x strictly feasible and x_k is large, then

$$x_1 \geq d_2 x_2^{\alpha_2},\ x_2 \geq d_3 x_3^{\alpha_3}, \ldots,\ x_{k-1} \geq d_k x_k^{\alpha_k}$$

where

$$2 \geq \alpha_2 \geq \frac{k}{k-1}, 2 \geq \alpha_3 \geq \frac{k-1}{k-2}, \ldots,\ 2 \geq \alpha_k \geq 2.$$

The d_j and α_j are constants that depend on the A_i, on B and $x_{k+1}, \ldots, x_m$ that we consider fixed.

Khachiyan type hierarchy in all strictly feasible SDPs.

Theorem 1 (Informal)

After a linear change of variables $x \leftarrow Mx$, if x strictly feasible and x_k is large, then

$$x_1 \geq d_2 x_2^{\alpha_2},\ x_2 \geq d_3 x_3^{\alpha_3}, \ldots,\ x_{k-1} \geq d_k x_k^{\alpha_k}$$

where

$$2 \geq \alpha_2 \geq \frac{k}{k-1}, 2 \geq \alpha_3 \geq \frac{k-1}{k-2}, \ldots,\ 2 \geq \alpha_k \geq 2.$$

The d_j and α_j are constants that depend on the A_i, on B and $x_{k+1}, \ldots, x_m$ that we consider fixed.

Khachiyan type hierarchy in all strictly feasible SDPs.

Assumptions are minimal.

Corollary

- In worst case (all $\alpha_j = 2$)

$$x_1 \geq \text{constant} \cdot x_k^{2^{k-1}}.$$

Corollary

- In worst case (all $\alpha_j = 2$)

$$x_1 \geq \text{constant} \cdot x_k^{2^{k-1}}.$$

- In best case (all α_j = lower bound)

$$x_1 \geq \text{constant} \cdot x_k^{k}.$$

Worst case example: Khachiyan SDP

$$\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & & x_3 \\ & & x_3 & & x_4 \\ & & & x_4 & \\ x_2 & x_3 & x_4 & & 1 \end{pmatrix} \succeq 0$$

Worst case example: Khachiyan SDP

$$\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & & x_3 \\ & & x_3 & & x_4 \\ & & & x_4 & \\ x_2 & x_3 & x_4 & & 1 \end{pmatrix} \succeq 0$$

- Subdeterminant with three red corners $\Rightarrow x_1 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

Worst case example: Khachiyan SDP

$$\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & & x_3 \\ & & x_3 & & x_4 \\ & & & x_4 & \\ x_2 & x_3 & x_4 & & 1 \end{pmatrix} \succeq 0$$

- Subdeterminant with three red corners $\Rightarrow x_1 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

Exponents are maximal.

Best case example: "Mild" SDP

$$\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix} \succeq 0$$

Best case example: "Mild" SDP

$$\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix} \succeq 0$$

- Subdeterminant with three red corners $\Rightarrow x_1 x_3 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 x_4 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

Best case example: “Mild” SDP

$$\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix} \succeq 0$$

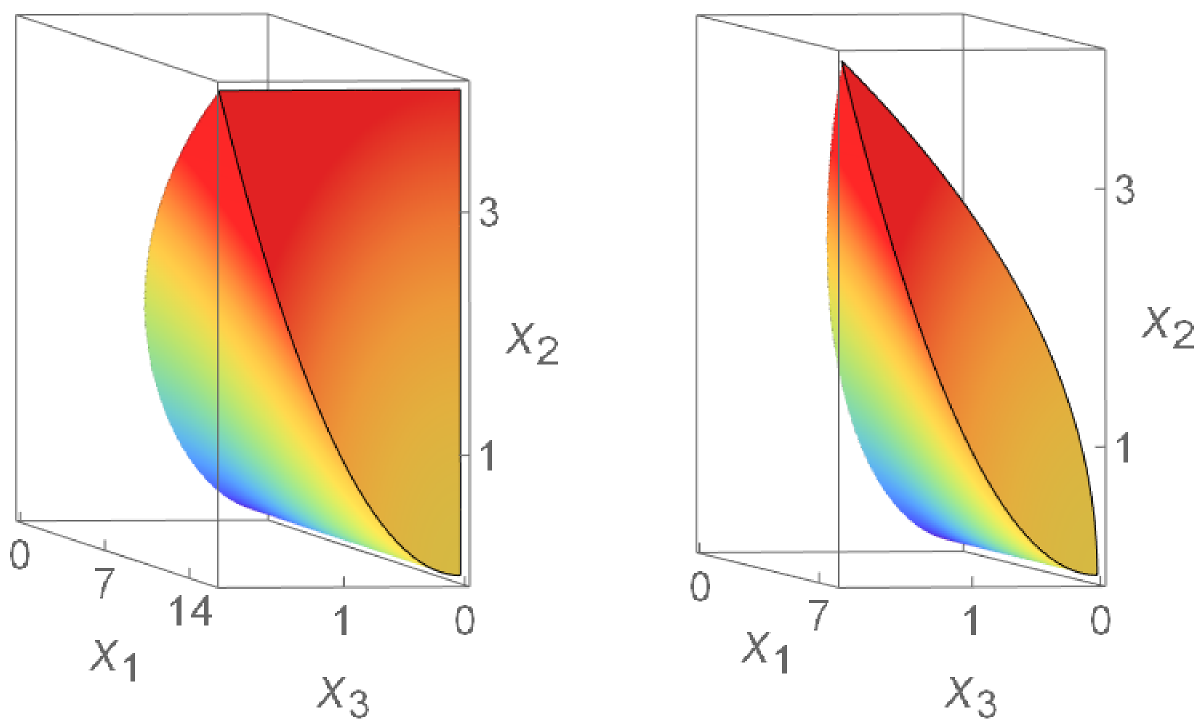
- Subdeterminant with three red corners $\Rightarrow x_1 x_3 \geq x_2^2$
- Subdeterminant with three blue corners $\Rightarrow x_2 x_4 \geq x_3^2$
- Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

From these we derive:

$$x_1 \geq x_2^{4/3},\ x_2 \geq x_3^{3/2},\ x_3 \geq x_4^2$$

Exponents are minimal.

Khachiyan vs Mild



- **Three variables,** $\mathbf{2 \geq x_3 \geq 0}$ **(normalization)**

Proof idea

(1) (**SDP**) $\longrightarrow$ **reformulate (change variables) to get** (**SDP'**)

Proof idea

(1) (**SDP**) $\longrightarrow$ **reformulate (change variables) to get** (**SDP'**)

(2) (**SDP'**) $\longrightarrow$ **messy quadratic inequalities such as**

$$(x_1 + 2x_2 + 5x_3)(x_4 + x_5) > (x_2 - 3x_6)^2$$

Proof idea

(1) (**SDP**) $\longrightarrow$ **reformulate (change variables) to get** (**SDP'**)

(2) (**SDP'**) $\longrightarrow$ **messy quadratic inequalities such as**

$$(x_1 + 2x_2 + 5x_3)(x_4 + x_5) > (x_2 - 3x_6)^2$$

(3) messy quadratic inequalities $\longrightarrow$ **cleaned up inequalities such as**

$$x_1 x_4 > \text{constant} \cdot x_2^2 \text{ if } x_k \text{ large}$$

Proof idea

(1) (SDP) $\longrightarrow$ reformulate (change variables) to get (SDP')

(2) (SDP') $\longrightarrow$ messy quadratic inequalities such as

$$(x_1 + 2x_2 + 5x_3)(x_4 + x_5) > (x_2 - 3x_6)^2$$

(3) messy quadratic inequalities $\longrightarrow$ cleaned up inequalities such as

$$x_1 x_4 > \text{constant} \cdot x_2^2 \text{ if } x_k \text{ large}$$

+ eliminate variables to get

$$x_j \geq \text{constant} \cdot x_{j+1}^{\alpha_{j+1}} \;\; \forall j$$

+ recursion to compute the α_{j+1}.

Reformulating (SDP) into (SDP')

The reformulated SDP looks like

$$x_1 \begin{pmatrix} \overbrace{\hphantom{I}}^{r_1} & \overbrace{\hphantom{0}}^{n-r_1} \\ I & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} \overbrace{\hphantom{\times}}^{r_1+\ldots+r_{i-1}} & \overbrace{\hphantom{I}}^{r_i} & \overbrace{\hphantom{0}}^{n-r_1-\ldots-r_i} \\ \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

$$+ \sum_{i=k+1}^{m} x_i A_i' + B' \succeq 0 \qquad \text{(SDP')}$$

with $r_1, \ldots, r_k > 0$.

Reformulating (SDP) into (SDP')

The reformulated SDP looks like

$$x_1 \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

$$+ \sum_{i=k+1}^{m} x_i A_i' + B' \succeq 0 \qquad \text{(SDP')}$$

with $r_1, \ldots, r_k > 0$.

To get this reformulation, we also used similarity transformations $T^\top () T$.

(SDP') → messy quadratic inequalities

$$\cdots + x_2 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{1} & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \end{pmatrix} + x_3 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{2} & \times & \textcircled{4} & \times & \times \\ \times & \times & \mathbf{1} & & & \\ \times & \textcircled{4} & & & & \\ \times & \times & & & & \\ \times & \times & & & & \end{pmatrix} + x_4 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{5} & \times & \textcircled{7} & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \textcircled{7} & \times & \textcircled{1} & & \\ \times & \times & & & & \\ \times & \times & & & & \end{pmatrix} + \cdots \succ \mathbf{0}$$

(SDP') → messy quadratic inequalities

$$\cdots + x_2 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{1} & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \end{pmatrix} + x_3 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{2} & \times & \textcircled{4} & \times & \times \\ \times & \times & \mathbf{1} & & & \\ \times & \textcircled{4} & & & & \\ \times & \times & & & & \\ \times & \times & & & & \end{pmatrix} + x_4 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{5} & \times & \textcircled{7} & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \textcircled{7} & \times & \textcircled{1} & & \\ \times & \times & & & & \\ \times & \times & & & & \end{pmatrix} + \cdots \succ 0$$

- **2 × 2 subdeterminant** →

$$(x_2 + 2x_3 + 5x_4 + \ldots)(x_4 + \ldots) > (4x_3 + 7x_4 + \ldots)^2$$

(SDP') → messy quadratic inequalities

$$\cdots + x_2 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{1} & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \end{pmatrix} + x_3 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{2} & \times & \textcircled{4} & \times & \times \\ \times & \times & \mathbf{1} & & & \\ \times & \textcircled{4} & & & & \\ \times & \times & & & & \\ \times & \times & & & & \end{pmatrix} + x_4 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{5} & \times & \textcircled{7} & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \textcircled{7} & \times & \textcircled{1} & & \\ \times & \times & & & & \\ \times & \times & & & & \end{pmatrix} + \cdots \succ \mathbf{0}$$

- 2×2 subdeterminant →

$$(x_2 + 2x_3 + 5x_4 + \ldots)(x_4 + \ldots) > (4x_3 + 7x_4 + \ldots)^2$$

- The $\ldots$ mean a combination of higher numbered variables.

(SDP') $\rightarrow$ messy quadratic inequalities

$$\cdots + x_2 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{1} & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \end{pmatrix} + x_3 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{2} & \times & \textcircled{4} & \times & \times \\ \times & \times & \mathbf{1} & & & \\ \times & \textcircled{4} & & & & \\ \times & \times & & & & \\ \times & \times & & & & \end{pmatrix} + x_4 \begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \textcircled{5} & \times & \textcircled{7} & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \textcircled{7} & \times & \textcircled{1} & & \\ \times & \times & & & & \\ \times & \times & & & & \end{pmatrix} + \cdots \succ \mathbf{0}$$

- $\mathbf{2 \times 2}$ subdeterminant $\rightarrow$

$$(x_2 + 2x_3 + 5x_4 + \ldots)(x_4 + \ldots) > (4x_3 + 7x_4 + \ldots)^2$$

- The $\ldots$ mean a combination of higher numbered variables.
- Cleaned up version (suppress higher numbered terms):

$$x_2 x_4 > \text{const} \cdot x_3^2, \quad \text{if } x_k \text{ is large}$$

How to compute the α_{j+1} in $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$?

$$\alpha_{j+1} = \begin{cases} 2 - \dfrac{1}{\alpha_{j+2} \ldots \alpha_{t_{j+1}}} \text{ if } t_{j+1} \leq k \\ 2 \text{ if } t_{j+1} = k+1 \end{cases}$$

for $j = 1, \ldots, k-1$.

How to compute the α_{j+1} in $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$?

$$\alpha_{j+1} = \begin{cases} 2 - \dfrac{1}{\alpha_{j+2} \ldots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\ 2 & \text{if } t_{j+1} = k+1 \end{cases}$$

for $j = 1, \ldots, k-1$.

Similar to continued fractions.

How to compute the α_{j+1} in $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$?

$$\alpha_{j+1} = \begin{cases} 2 - \dfrac{1}{\alpha_{j+2} \ldots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\ 2 & \text{if } t_{j+1} = k+1 \end{cases}$$

for $j = 1, \ldots, k-1$.

Similar to continued fractions.

Here

$$t_{j+1} = \text{index of a rightmost block with } x_{j+1}$$

Shift x_{j+1} to right $\Rightarrow$ t_{j+1} increases.

How to compute the α_{j+1} in $x_j \geq \mathrm{const} \cdot x_{j+1}^{\alpha_{j+1}}$?

$$\alpha_{j+1} = \begin{cases} 2 - \dfrac{1}{\alpha_{j+2} \ldots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\ 2 & \text{if } t_{j+1} = k+1 \end{cases}$$

for $j = 1, \ldots, k-1$.

Similar to continued fractions.

Here

$$t_{j+1} = \text{index of a rightmost block with } x_{j+1}$$

Shift x_{j+1} **to right** $\Rightarrow$ t_{j+1} **increases.**

$\Rightarrow$ α_{j+1} **increases.**

Example

$$\underbrace{\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha = (4/3,\ 3/2,\ 2)}$$

Example

$$\underbrace{\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha = (4/3,\ 3/2,\ 2)} \rightarrow \underbrace{\begin{pmatrix} x_1 & & & x_2 & \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ x_2 & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha = (5/3,\ 3/2,\ 2)}$$

Example

$$\underbrace{\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(4/3,\ 3/2,\ 2)} \rightarrow \underbrace{\begin{pmatrix} x_1 & & & x_2 & \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ x_2 & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(5/3,\ 3/2,\ 2)} \rightarrow \underbrace{\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ x_2 & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(2,\ 3/2,\ 2)}$$

Example

$$\underbrace{\begin{pmatrix} x_1 & & x_2 & & \\ & x_2 & & x_3 & \\ x_2 & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(4/3,\ 3/2,\ 2)} \to \underbrace{\begin{pmatrix} x_1 & & & x_2 & \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ x_2 & x_3 & & x_4 & \\ & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(5/3,\ 3/2,\ 2)} \to \underbrace{\begin{pmatrix} x_1 & & & & x_2 \\ & x_2 & & x_3 & \\ & & x_3 & & x_4 \\ & x_3 & & x_4 & \\ x_2 & & x_4 & & 1 \end{pmatrix}}_{\alpha\,=(2,\ 3/2,\ 2)}$$

In other words

$$x_1 \geq x_2^{4/3} \to x_1 \geq x_2^{5/3} \to x_1 \geq x_2^2$$

Connection to Fourier-Motzkin elimination

- **We are eliminating variables to get to $x_j \geq \text{const}\, x_{j+1}^{\alpha_{j+1}}$.**
- **This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$.**

Connection to Fourier-Motzkin elimination

- We are eliminating variables to get to $x_j \geq \text{const}\, x_{j+1}^{\alpha_{j+1}}$.
- This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$.
- Example

$$
\begin{aligned}
x_1 x_3 &\geq x_2^2 \\
x_2 x_4 &\geq x_3^2 \\
x_3 &\geq x_4^2
\end{aligned}
$$

Connection to Fourier-Motzkin elimination

- We are eliminating variables to get to $x_j \geq x_{j+1}^{\alpha_{j+1}}$.
- This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$.
- Example

$$
\begin{aligned}
x_1 x_3 &\geq x_2^2 & & & y_1 + y_3 &\geq 2y_2 \\
x_2 x_4 &\geq x_3^2 & &\rightarrow & y_2 + y_4 &\geq 2y_3 \\
x_3 &\geq x_4^2 & & & y_3 &\geq 2y_4
\end{aligned}
$$

Connection to Fourier-Motzkin elimination

- We are eliminating variables to get to $x_j \geq x_{j+1}^{\alpha_{j+1}}$.
- This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$.
- Example

$$\begin{aligned} x_1 x_3 &\geq x_2^2 & \qquad y_1 + y_3 &\geq 2y_2 \\ x_2 x_4 &\geq x_3^2 & \rightarrow \quad y_2 + y_4 &\geq 2y_3 \\ x_3 &\geq x_4^2 & y_3 &\geq 2y_4 \end{aligned}$$

- Add $1/2$ times the last to the middle:

$$y_2 \geq \tfrac{3}{2} y_3$$

Connection to Fourier-Motzkin elimination

- We are eliminating variables to get to $x_j \geq x_{j+1}^{\alpha_{j+1}}$.
- This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$.
- Example

$$\begin{aligned} x_1 x_3 &\geq x_2^2 & \qquad & & y_1 + y_3 &\geq 2y_2 \\ x_2 x_4 &\geq x_3^2 & \qquad & \rightarrow & y_2 + y_4 &\geq 2y_3 \\ x_3 &\geq x_4^2 & \qquad & & y_3 &\geq 2y_4 \end{aligned}$$

- Add $1/2$ times the last to the middle:

$$x_2 \geq x_3^{3/2} \leftarrow y_2 \geq \frac{3}{2} y_3$$

Do we need the change of variables $x \leftarrow Mx$?

- **In general, yes: such an operation may mess up even** (**Khachiyan**).
- **So, we may need such an operation $x \leftarrow M^{-1}x$ to unmess it.**
- **But, sometimes we don't.**

When we do not even need a change of variables, part 1

Want to minimize $f(x) =$ univariate degree $2n$ polynomial.

Rewrite as SDP, using sum-of-squares technique, look at dual (show the case $n = 3$)

$$y_6 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + y_4 \begin{pmatrix} 0 & & 1 & \\ & 1 & & \\ 1 & & 0 & \\ & & & 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} + \cdots \succeq 0$$

When we do not even need a change of variables, part 1

Want to minimize $f(x) =$ univariate degree $2n$ polynomial.

Rewrite as SDP, using sum-of-squares technique, look at dual (show the case $n = 3$)

$$y_6 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + y_4 \begin{pmatrix} 0 & & 1 & \\ & 1 & & \\ 1 & & 0 & \\ & & & 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} + \cdots \succeq 0$$

Exactly in the form of (SDP'), without a change of variables.

When we do not even need a change of variables, part 1

Want to minimize $f(x) =$ univariate degree $2n$ polynomial.

Rewrite as SDP, using sum-of-squares technique, look at dual (show the case $n = 3$)

$$y_6 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + y_4 \begin{pmatrix} 0 & & 1 & \\ & 1 & & \\ 1 & & 0 & \\ & & & 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} + \cdots \succeq 0$$

Exactly in the form of (SDP'), without a change of variables.

Corollary: $y \in \mathbb{R}^{2n}$ feasible $\Rightarrow$

$$y_{2n} \geq y_{2n-2}^{1+1/(n-1)}, \; y_{2n-2} \geq y_{2n-4}^{1+1/(n-2)}, \ldots$$

$$y_{2n} \geq y_2^n.$$

When we do not even need a change of variables, part 2

O' Donnell, 2017 We want to certify that a polynomial

$$p(x, y) = x_1 + \cdots + x_n - 2y_1 \geq 0$$

for all $(x, y) \in K$, where K is a simple set.

When we do not even need a change of variables, part 2

O' Donnell, 2017 We want to certify that a polynomial

$$p(x, y) = x_1 + \cdots + x_n - 2y_1 \geq 0$$

for all $(x, y) \in K$, where K is a simple set.

Resulting SDP:

$$u_1 E_{11} + \sum_{i=2}^{n} u_i \big(E_{ii} - E_{i-1,n+i-1}\big) + B \succeq 0.$$

Here E_{ij} is the (i, j) unit matrix.

Exactly in the form of (SDP') ! It yields essentially Khachiyan's example.

Certifying exponential size solutions in polynomial space, without computing them

Revisiting the reformulated problem:

$$x_1 \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

$$+ \sum_{i=k+1}^{m} x_i A_i' + B' \succeq 0$$

(**SDP'**)

Certifying exponential size solutions in polynomial space, without computing them

Revisiting the reformulated problem:

$$x_1 \begin{pmatrix} \overbrace{\boldsymbol{I}}^{r_1} & \overbrace{\mathbf{0}}^{n-r_1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & \boldsymbol{I} & \mathbf{0} \\ \times & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$+ \sum_{i=k+1}^{m} x_i A_i' + B' \succeq \mathbf{0}$$

(**SDP'**)

Suppose we have $x_{k+1}, \ldots, x_m$ **s.t.** $\exists\; x_1, \ldots, x_k$ **so this problem is strictly feasible.**

Certifying exponential size solutions in polynomial space, without computing them

Revisiting the reformulated problem:

$$x_1 \begin{pmatrix} \overbrace{\boldsymbol{I}}^{r_1} & \overbrace{\mathbf{0}}^{n-r_1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & \boldsymbol{I} & \mathbf{0} \\ \times & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$+ \sum_{i=k+1}^{m} x_i A_i' + B' \succeq \mathbf{0}$$

(**SDP'**)

Suppose we have $x_{k+1}, \ldots, x_m$ **s.t.** $\exists\, x_1, \ldots, x_k$ **so this problem is strictly feasible.**

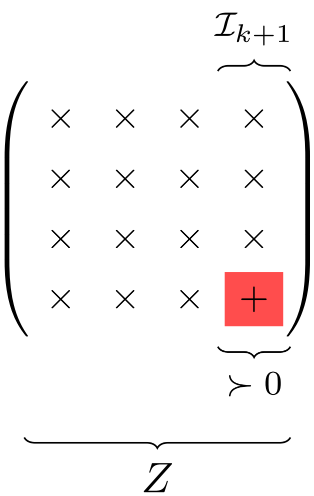
Then we can prove that $x_1, \ldots, x_k$ **exist without having to compute them.**

Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^m x_i A_i' + B'$ positive definite. Start with $Z := \sum_{i=k+1}^m x_i A_i' + B'$

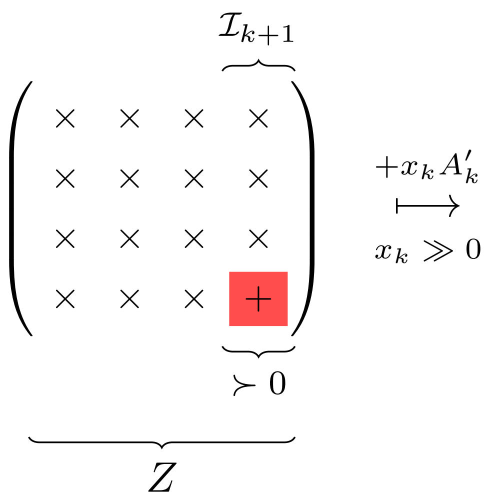
Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^{m} x_i A'_i + B'$ positive definite. Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.



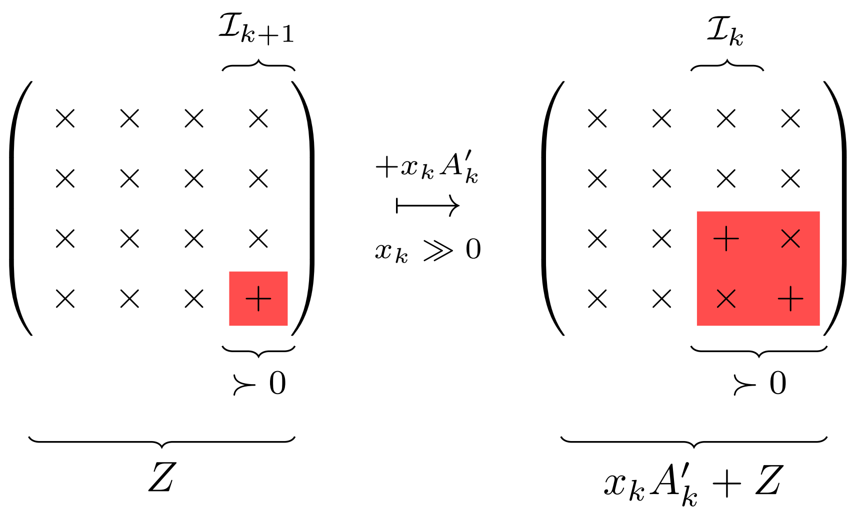
Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^m x_i A_i' + B'$ positive definite. Start with $Z := \sum_{i=k+1}^m x_i A_i' + B'$.



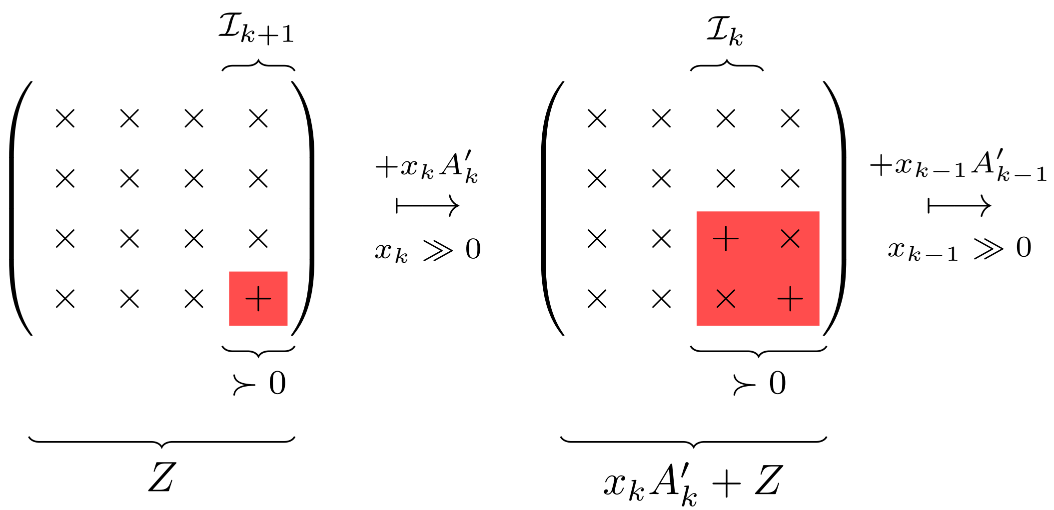
Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^{m} x_i A'_i + B'$ positive definite. Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.



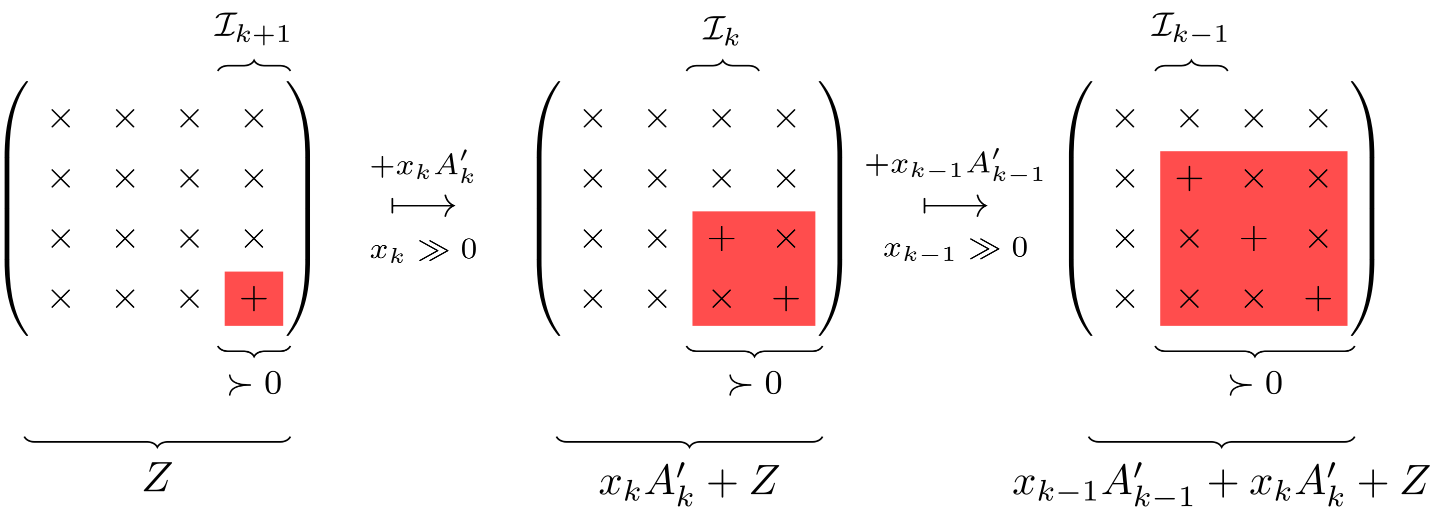
Verifying that $x_1, \dots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^{m} x_i A'_i + B'$ positive definite. Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.



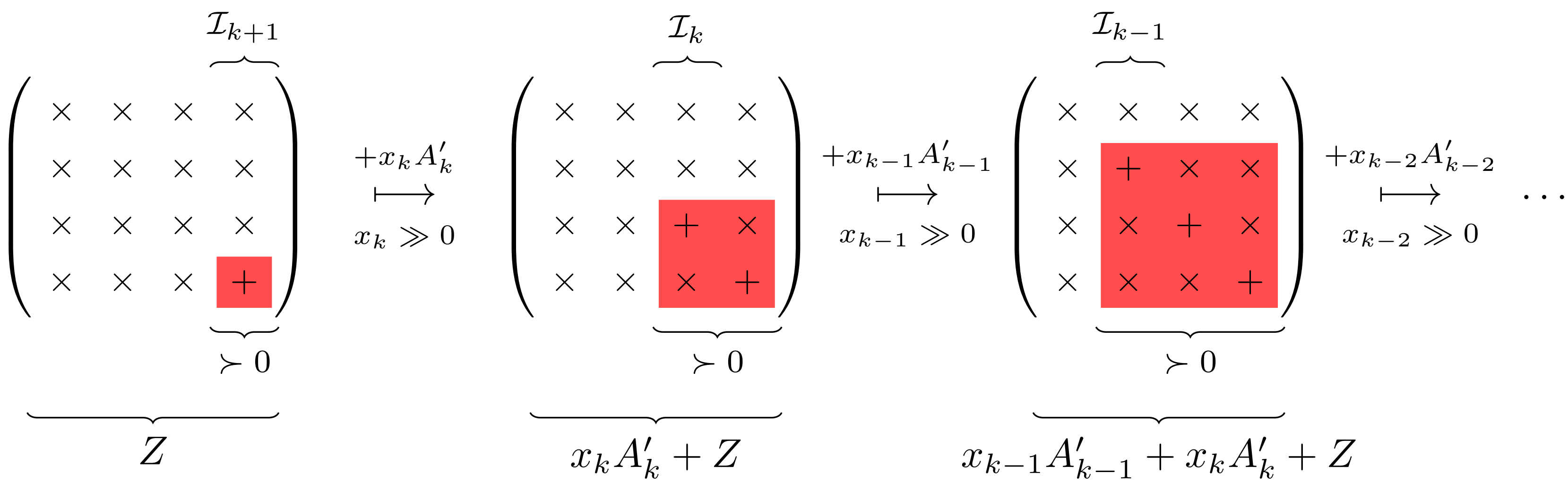
Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^m x_i A_i' + B'$ positive definite. Start with $Z := \sum_{i=k+1}^m x_i A_i' + B'$.



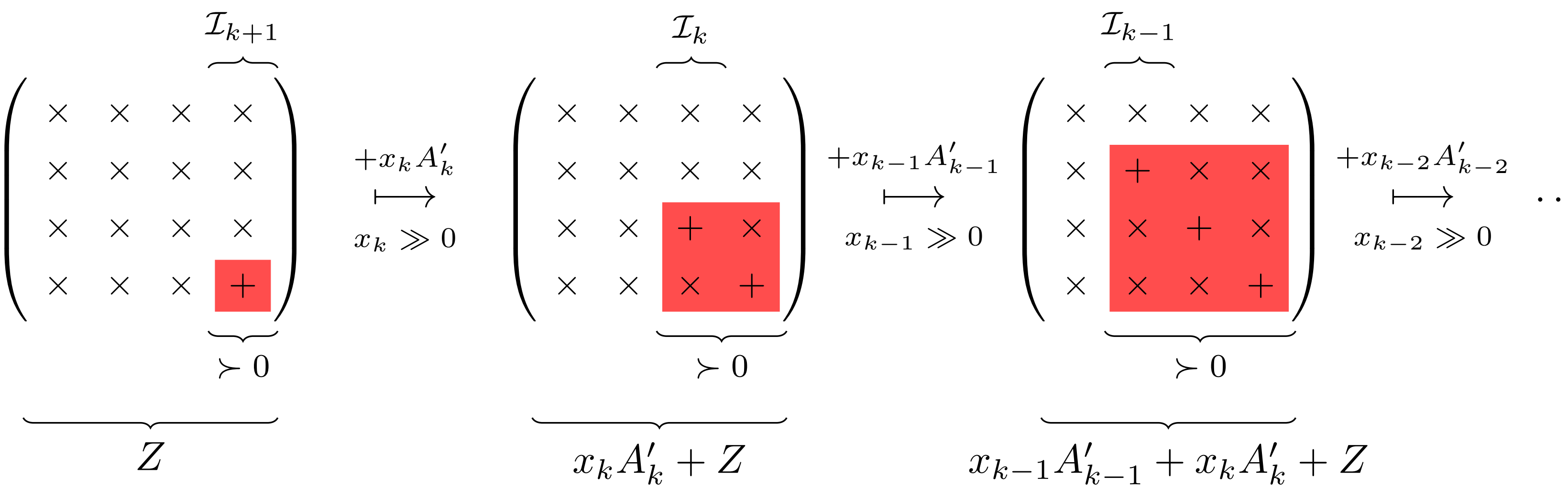
Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^{m} x_i A'_i + B'$ positive definite. Start with $Z := \sum_{i=k+1}^{m} x_i A'_i + B'$.



Verifying that $x_1, \ldots, x_k$ exist, without computing them

Could compute them in reverse order, to make larger and larger lower right corners of $\sum_{i=1}^m x_i A_i' + B'$ positive definite. Start with $Z := \sum_{i=k+1}^m x_i A_i' + B'$



Question: Are all SDPs with large solutions in this regularized form (maybe after a similarity transformation)?

Conclusion

- **Exponential size solutions in SDP, going back to famous Khachiyan example.**
- **Khachiyan type hierarchy among leading variables in every strictly feasible SDP (after linear change of variables)**
- **Formulas to compute the exponents (like continued fractions)**

Conclusion

- **Exponential size solutions in SDP, going back to famous Khachiyan example.**
- **Khachiyan type hierarchy among leading variables in every strictly feasible SDP (after linear change of variables)**
- **Formulas to compute the exponents (like continued fractions)**
- **Partial answer to: how to represent exponential size solutions in polynomial space?**
- **Every known SDP with large solutions is in our normal form (SDP').**
- **Paper:** https://arxiv.org/abs/2103.00041

Thank you!