How do Exponential Size Solutions Arise in Semidefinite Programming?

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Joint work with Alex Touzov  
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Linear Programming (LP) feasibility

\[ \exists \ x \ s.t. \quad Ax \geq b \quad (LP) \]

Here

- \( A \in \mathbb{Z}^{m\times n}, \ b \in \mathbb{Z}^m. \)
- \((LP)\) feasible \( \Rightarrow \exists \) feasible rational \( \bar{x} \) with size (\# of bits needed to describe it) is
  \[ \leq 2n^2(\log n)(\log L) \]
  where \( L = \) largest entry in \( A, b. \)
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  where \(L = \) largest entry in \(A, b.\)
- \(\rightarrow\) To solve LP feasibility in poly time, we find a solution \(\bar{x}\.\)
Semidefinite Programming (SDP) feasibility

\[ \exists \exists x \text{ s.t. } \sum_{i=1}^{m} x_i A_i + B \succeq 0 \]  \hspace{1cm} (SDP)

Here

- \( A_i, B \) are symmetric matrices,
- \( S \succeq 0 \) means that \( S \) is symmetric positive semidefinite (psd).
- Far reaching generalization of LP.
In SDPs exponential size solutions are unavoidable

- **Khachiyan example**

  \[ x_1 \geq x_2^2, \ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2, \ x_m \geq 2. \]  

  (Khachiyan)

- **x feasible \(\Rightarrow\) \( x_1 \geq 2^{2m-1}. \)**
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- Khachiyan example
  \[ x_1 \geq x_2^2, \ x_2 \geq x_3^2, \ldots, x_{m-1} \geq x_m^2, \ x_m \geq 2. \] (Khachiyan)

- \( x \) feasible \( \Rightarrow \) \( x_1 \geq 2^{2^{m-1}} \).

- Size of \( x \geq \log 2^{2^{m-1}} = 2^{m-1} \).

- Can be written as SDP:
  \[ x_i \geq x_{i+1}^2 \Leftrightarrow \begin{pmatrix} x_i & x_{i+1} \\ x_{i+1} & 1 \end{pmatrix} \succeq 0 \forall i. \]
$$x_1 \geq x_2^2, \ x_2 \geq x_3^2, \ 2 \geq x_3 \geq 0$$ (1)
Is \textbf{(SDP)} feasibility in \textit{P}?

- Major open problem
- Open even for QCQPs
Is (SDP) feasibility in P?

- Major open problem
- Open even for QCQPs
- Exponential size solutions are a major obstacle
- How to prove in polynomial time that a possibly exponential size solution exists?
Question 1

• Can we represent such large solutions in polynomial space?
• (Khachiyan) gives hope: no need to write out $2^{2m-1}$ to convince ourselves that $x_1 = 2^{2m-1}$ is feasible.
Question 2
Are large solutions common in SDPs?

Seemingly no, since:
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- they do not come up in LPs, or “typical” SDPs.
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- we may eliminate them even in *(Khachiyan)* by a very slight change, as:
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Are large solutions common in SDPs?

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• they do not come up in LPs, or “typical” SDPs.
• we may eliminate them even in \((\text{Khachiyan})\) by a very slight change, as:

(1) replace

\[ x_m \geq 2 \rightarrow x_m \geq 2 + x_{m+1} \]

where \(x_{m+1}\) is a new variable

\[ \rightarrow x_1 \text{ does not have to be large anymore.} \]
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  \[ x \leftarrow Gx \]
  where \( G \) is random dense matrix.
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(2) by linear change of variables:
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  where \( G \) is random dense matrix.
  \( \rightarrow \) (Khachiyan) becomes a big mess.
  \( \rightarrow \) Apparent common consent: large variables in SDPs are rare.
However: Main result (informal)

• We can “untangle” any strictly feasible SDP and make it into a Khachiyan type SDP.
Background

- $k :=$ singularity degree of $\{ Y \succeq 0 : A_i \bullet Y = 0 \forall i \}$.
- minimum number of facial reduction steps to certify maximum rank psd matrix
- $k \leq 1$ when (SDP) is an LP.
Background

- \( k := \) singularity degree of \( \{ Y \succeq 0 : A_i \bullet Y = 0 \forall i \} \).

- minimum number of facial reduction steps to certify maximum rank psd matrix

- \( k \leq 1 \) when (SDP) is an LP.

- We assume that (SDP) is strictly feasible, i.e., \( \exists x \) s.t.

\[ \sum_{i=1}^{m} x_i A_i + B \succ 0. \]
Theorem 1 (Informal)

After a linear change of variables $x \leftarrow Mx$, if $x$ strictly feasible and $x_k$ is large, then

$$x_1 \geq d_2 x_2^{\alpha_2}, \ x_2 \geq d_3 x_3^{\alpha_3}, \ldots, \ x_{k-1} \geq d_k x_k^{\alpha_k}$$

where

$$2 \geq \alpha_2 \geq \frac{k}{k-1}, \ 2 \geq \alpha_3 \geq \frac{k-1}{k-2}, \ldots, \ 2 \geq \alpha_k \geq 2.$$
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The $d_j$ and $\alpha_j$ are constants that depend on the $A_i$, on $B$ and $x_{k+1}, \ldots, x_m$ that we consider fixed.

Khachiyan type hierarchy in all strictly feasible SDPs.

Assumptions are minimal.
Corollary

• In worst case (all $\alpha_j = 2$)
  
  \[ x_1 \geq \text{constant} \cdot x_k^{2^{k-1}}. \]

• In best case (all $\alpha_j = \text{lower bound}$)
  
  \[ x_1 \geq \text{constant} \cdot x_k^k. \]
Worst case example: Khachiyan SDP

\[
\begin{pmatrix}
  x_1 & x_2 \\
  x_2 & x_3 \\
  x_3 & x_4 \\
  x_4 & 1
\end{pmatrix} \succeq 0
\]

- Subdeterminant with three red corners \( \Rightarrow x_1 \geq x_2^2 \)
- Subdeterminant with three blue corners \( \Rightarrow x_2 \geq x_3^2 \)
- Subdeterminant with three green corners \( \Rightarrow x_3 \geq x_4^2 \)

Exponents are maximal.
Best case example: “Mild” SDP

\[
\begin{pmatrix}
  x_1 & x_2 \\
  x_2 & x_3 \\
  x_2 & x_3 & x_4 \\
  x_3 & x_4 \\
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• Subdeterminant with three red corners \( \Rightarrow x_1 x_3 \geq x_2^2 \)
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• Subdeterminant with three red corners $\Rightarrow x_1 x_3 \geq x_2^2$
• Subdeterminant with three blue corners $\Rightarrow x_2 x_4 \geq x_3^2$
• Subdeterminant with three green corners $\Rightarrow x_3 \geq x_4^2$

From these we derive:

\[
x_1 \geq x_2^{4/3}, \ x_2 \geq x_3^{3/2}, \ x_3 \geq x_4^2
\]

Exponents are minimal.
Khachiyon vs Mild

- Three variables, $2 \geq x_3 \geq 0$ (normalization)
Proof idea

(SDP) $\rightarrow$ reformulate (change variables)
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$\rightarrow$ messy quadratic inequalities such as

$$(x_1 + 2x_2 + 5x_3)(x_4 + x_5) \geq (x_2 - 3x_6)^2$$
Proof idea

\[(\text{SDP}) \quad \rightarrow \quad \text{reformulate (change variables)}\]

\[\rightarrow \text{messy quadratic inequalities such as} \quad (x_1 + 2x_2 + 5x_3)(x_4 + x_5) \geq (x_2 - 3x_6)^2\]

\[\rightarrow \text{cleaned up quadratic inequalities} \quad x_j \geq \text{constant} \cdot x_j^{\alpha j+1}\]
Proof idea

(SDP) $\rightarrow$ reformulate (change variables)

$\rightarrow$ messy quadratic inequalities such as

$$(x_1 + 2x_2 + 5x_3)(x_4 + x_5) \geq (x_2 - 3x_6)^2$$

$\rightarrow$ cleaned up quadratic inequalities

$$x_j \geq \text{constant} \cdot x_{j+1}^{\alpha_{j+1}}$$

+ recursion to compute the $\alpha_{j+1}$. 
The reformulated SDP

... looks like

\[ x_1 \begin{pmatrix} r_1 & n-r_1 \\ I & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=2}^{k} x_i \begin{pmatrix} r_1+\ldots+r_{i-1} & r_i & n-r_1-\ldots-r_i \\ \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \]

\[ + \sum_{i=k+1}^{m} x_i A'_i + B' \succeq 0 \]

with \( r_1, \ldots, r_k > 0 \).

To get this reformulation, we also used similarity transformations \( T^\top()T \).
Reformulated SDP $\rightarrow$ messy polynomials

Here $\bullet$ means a nonzero block.
Reformulated SDP $\rightarrow$ messy polynomials

Here $\bullet$ means a nonzero block.
Submatrix with $\bullet$ corners:

$$
\begin{pmatrix}
x_j + \ldots & \text{const } x_{j+1} + \ldots \\
\text{const } x_{j+1} + \ldots & x_{t_{j+1}} + \ldots
\end{pmatrix} \succ 0.
$$

where $\cdots = \text{higher numbered variables}$
Determinant $\rightarrow$ messy polynomial.
How to compute the $\alpha_{j+1}$ in $x_j \geq \text{const} \cdot x_{j+1}^{\alpha_{j+1}}$?

$$
\alpha_{j+1} = \begin{cases} 
2 - \frac{1}{\alpha_{j+2} \cdots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\
2 & \text{if } t_{j+1} = k + 1
\end{cases}
$$

for $j = 1, \ldots, k - 1$. 
How to compute the $\alpha_{j+1}$ in $x_j \geq \text{const} \cdot x_j^{\alpha_{j+1}}$?

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Similar to continued fractions.
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\end{cases}$$

for $j = 1, \ldots, k - 1$.

Similar to continued fractions.

Here

$$t_{j+1} = \text{index of a rightmost block where } x_{j+1} \text{ shows up.}$$

We shift $x_{j+1}$ to right $\Rightarrow \alpha_{j+1}$ increases.
Example

\[
\begin{pmatrix}
 x_1 & x_2 \\
 & x_2 & x_3 \\
 & x_2 & x_3 & x_4 \\
 & x_3 & x_4 \\
 & x_4 & 1
\end{pmatrix}
\]

\[
\alpha = (4/3, \frac{3}{2}, 2)
\]
Example

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\begin{pmatrix}
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\end{pmatrix}
\quad \xRightarrow{\alpha = (2, 3/2, 2)}
\]
Connection to Fourier-Motzkin elimination

- We are eliminating variables to get to $x_j \geq x_j^{\alpha j+1}$.
- This process can be viewed as Fourier-Motzkin elimination via $y_i := \log x_i$. 
Connection to Fourier-Motzkin elimination

• We are eliminating variables to get to \( x_j \geq x_{\alpha j+1}^{j+1} \).

• This process can be viewed as Fourier-Motzkin elimination via \( y_i := \log x_i \).

• Example

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\begin{align*}
    x_1x_3 & \geq x_2^2 \\
    x_2x_4 & \geq x_3^2 \\
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- Example

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  \begin{align*}
  x_1x_3 & \geq x_2^2 & y_1 + y_3 & \geq 2y_2 \\
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x_2x_4 & \geq x_3^2 & \rightarrow y_2 + y_4 & \geq 2y_3 \\
x_3 & \geq x_4^2 & y_3 & \geq 2y_4
\end{align*}
\]

• Add 1/2 times the last to the middle:

\[
x_2 \geq x_3^{3/2} \quad \leftarrow \quad y_2 \geq \frac{3}{2}y_3
\]
Do we need the change of variables \( x \leftarrow Mx \)?

- In general, yes: such an operation may mess up even (Khachiyan).
- So, we may need such an operation \( x \leftarrow M^{-1}x \) to unmess it.
- But, sometimes we don’t.
Poly opt. SDPs do not need reformulation

We want to minimize $f(x) = \text{univariate degree } 2n \text{ polynomial}.$

Rewrite as SDP, dual matrix has Hankel structure ($y_0 = 1$):

$$
\begin{pmatrix}
  y_0 & y_1 & y_2 & \cdots & y_n \\
  y_1 & y_2 & \cdots & y_{n+1} \\
  y_2 & \cdots & y_{n+2} \\
  \vdots & \ddots & \vdots \\
  y_n & y_{n+1} & y_{n+2} & \cdots & y_{2n}
\end{pmatrix} \succeq 0. \quad \text{(Poly-SDP)}
$$
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Exactly in the form we want without a change of variables.
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  y_2 & \cdots & y_{n+2} \\
  \vdots & \ddots & \ddots \\
  y_n & y_{n+1} & y_{n+2} & \cdots & y_{2n}
\end{pmatrix} \succeq 0. \quad (\text{Poly-SDP})
\]

Exactly in the form we want without a change of variables.

Corollary: \( y \in \mathbb{R}^{2n} \text{ feasible (Poly-SDP)} \Rightarrow \)

\[
y_{2n} \geq y_{2n-2}^{1+1/(n-1)}, \quad y_{2n-2} \geq y_{2n-4}^{1+1/(n-2)}, \ldots
\]

\[
y_{2n} \geq y_2^n.
\]
Conclusion

• Exponential size solutions in SDP, going back to famous Khachiyan example.
• Khachiyan type hierarchy among leading variables (after linear change of variables)
• Formulas to compute the exponents.
Conclusion

- Exponential size solutions in SDP, going back to famous Khachiyan example.
- Khachiyan type hierarchy among leading variables in every strictly feasible SDP (after linear change of variables)
- Formulas to compute the exponents.
- Connection to: continued fractions and Fourier-Motzkin elimination
- Assumptions we make are minimal.
- Paper coming very soon.
Thank you!