On positive duality gaps in semidefinite programming

Gábor Pataki

UNC Chapel Hill

Talk at RICAM, 2019

A pair of Semidefinite Programs (SDP)

$$
(P) \quad \begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array}
\qquad\qquad
\begin{array}{ll} \inf_Y & B \bullet Y \\ & Y \succeq 0 \\ & A_i \bullet Y = c_i \, \forall i. \end{array} \quad (D)
$$

Here

- A_i, B are symmetric matrices, $c, x \in \mathbb{R}^m$.
- $A \preceq B$ means that $B - A$ is symmetric positive semidefinite (psd).
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

SDP duality

$$(P) \quad \begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array} \qquad \begin{array}{ll} \inf_Y & B \bullet Y \\ & Y \succeq 0 \\ & A_i \bullet Y = c_i \, \forall i. \end{array} \quad (D)$$

SDP duality

$$
(P)\quad \begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array}
\qquad\qquad
\begin{array}{ll} \inf_Y & B \bullet Y \\ & Y \succeq 0 \\ & A_i \bullet Y = c_i \, \forall i. \end{array} \quad (D)
$$

Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

SDP duality

$$
(P) \quad \begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array}
\qquad\qquad
\begin{array}{ll} \inf_Y & B \bullet Y \\ & Y \succeq 0 \\ & A_i \bullet Y = c_i \, \forall i. \end{array} \quad (D)
$$

Easy: If x and Y are feasible, then $c^T x \le B \bullet Y$.

Ideally: $\exists x^*, \exists Y^* : c^T x^* = B \bullet Y^*$.

SDP duality

$$\begin{array}{llll} & \sup_x \ c^T x & \inf_Y \ B \bullet Y & \\ (P) & s.t. \ \sum_{i=1}^m x_i A_i \preceq B & Y \succeq 0 & (D) \\ & & A_i \bullet Y = c_i \, \forall i. & \end{array}$$

Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

Ideally: $\exists x^*, \exists Y^* : c^T x^* = B \bullet Y^*$.

But: pathologies occur, as nonattainment, positive gaps.

SDP duality

$$
(P) \quad \begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array} \qquad\qquad \begin{array}{ll} \inf_Y & B \bullet Y \\ & Y \succeq 0 \\ & A_i \bullet Y = c_i \, \forall i. \end{array} \quad (D)
$$

Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

Ideally: $\exists x^*, \exists Y^* : c^T x^* = B \bullet Y^*$.

But: pathologies occur, as nonattainment, positive gaps.

$\rightarrow$ in such cases we cannot certify optimality.

Example: positive duality gap

Primal:

$$\sup \; x_2$$

$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example: positive duality gap

Primal:

$$\sup \; x_2$$
$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 = 0$ **identically.**

Example: positive duality gap

Primal:

$$\sup \ x_2$$
$$s.t. \ x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 = 0$ identically $\Rightarrow$ primal $= 0$.

Example: positive duality gap

Primal:

$$\sup \; x_2$$

$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Dual matrix $Y \succeq 0$

Example: positive duality gap

Primal:

$$\sup \; x_2$$
$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow \; y_{11} = 0 \; \Rightarrow Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{23} & y_{33} \end{pmatrix}$

Example: positive duality gap

Primal:

$$\sup \; x_2$$
$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow \; y_{11} = 0 \; \Rightarrow Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{23} & y_{33} \end{pmatrix}$

Example: positive duality gap

Primal:

$$\sup \; x_2$$

$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow \; y_{11} = 0 \; \Rightarrow Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{23} & y_{33} \end{pmatrix}$

2nd dual constraint $\Rightarrow y_{22} = 1 \Rightarrow \text{dual opt} = 1$

Example: positive duality gap

Primal:

$$\sup \ x_2$$
$$s.t. \ x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Looks quite odd: x_1 "only exists" to create a zero block in the dual matrix.

Positive gaps

- **Maybe the "worst/most interesting" pathology.**
- **Solvers fail, or report a wrong solution.**
- **Good model of positive gaps in more general convex programs**

Literature

- Pathological semidefinite systems P, 2011 –
  "Bad semidefinite programs: they all look the same"
- It does not distinguish among bad objective functions.

Literature

- Pathological semidefinite systems P, 2011 –
  "Bad semidefinite programs: they all look the same"
- It does not distinguish among bad objective functions.
- Positive gaps related to complementarity in homogeneous systems: Tuncel, Wolkowicz, 2012
- Weak infeasibility: Lourenco, Muramatsu, Tsuchiya, 2014
- Infeasibility, weak infeasibility: Liu, P 2015, 2017

Literature: how to solve some pathological SDPs

- **Facial reduction of Borwein-Wolkowicz, Waki-Muramatsu, Pataki: implemented by Permenter, Parrilo 2014; Permenter,Friberg,Andersen 2015**

- **Very simple facial reduction (just inspect the constraints): Zhu, P, Tran Dinh (Sieve-SDP) 2017**

- **SPECTRA, exact arithmetic SDP solver Henrion-Naldi-El Din 2016**

- **Douglas-Rachford splitting: Liu, Ryu, Yin 2017**

- **Homotopy method Hauenstein, Liddell, Zhang 2018**

Main ideas

- **Look at small instances.**
- **Proposition: positive gap $\Rightarrow m \geq 2$.**
- **Fully characterize the $m = 2$ case.**

Main ideas

- **Look at small instances.**

  Proposition: positive gap $\Rightarrow m \geq 2$.

- **Fully characterize the $m = 2$ case.**

- **Show that the $m = 2$ case sheds light on larger m.**

- **Precisely: the structure that causes positive gap when $m = 2$ often does the same when $m > 2$.**

Main ideas

- **Look at small instances.**

  Proposition: positive gap $\Rightarrow m \geq 2$.

- **Fully characterize the $m = 2$ case.**

- **Show that the $m = 2$ case sheds light on larger m.**

- **Precisely: the structure that causes positive gap when $m = 2$ does the same in many cases even if $m > 2$.**

- **Reformulate**

- **Borrow ideas from linear system of equations:**
  to show $Ax = b$ is infeasible, we create an equation $\langle 0, x \rangle = 1$.

Recall: a pair of Semidefinite Programs (SDPs)

$$(P) \quad \begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array} \qquad \begin{array}{ll} \inf_Y & B \bullet Y \\ & Y \succeq 0 \\ & A_i \bullet Y = c_i \, \forall i. \end{array} \quad (D)$$

Reformulations of $(P) - (D)$ are obtained by

- Choose T invertible, and

$$B \leftarrow T^T B T,\ A_i \leftarrow T^T A_i T\ \forall i.$$

- Elementary row operations on (D) : e.g., exchange two constraints $A_i \bullet Y = c_i$ and $A_j \bullet Y = c_j$.

- Choose $\mu \in \mathbb{R}^m$ and

$$B \leftarrow B + \sum_{i=1}^{m} \mu_i A_i.$$

Reformulations preserve positive gaps (if any).

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$\sup \; c_2' x_2$$

$$\text{s.t. } x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array}\right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right)$$

where $\Lambda \succ 0, \; M \neq 0, \; c_2' > 0, s \geq 0.$

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$\sup \ c_2' x_2$$

$$s.t. \ x_1 \begin{pmatrix} \Lambda & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} + x_2 \begin{pmatrix} \times & \times & \times & M \\ \times & \Sigma & & \\ \times & & -I_s & \\ M^T & & & \end{pmatrix} \preceq \begin{pmatrix} I_p & & & \\ & I_{r-p} & & \\ & & & \\ & & & \end{pmatrix}$$

where $\Lambda \succ 0, \ M \neq 0, \ c_2' > 0, s \geq 0.$

Proof of $\Leftarrow$

This is the easy direction.

Essentially reuse the argument from before.

Suppose $m = 2$.

Then positive gap $\Leftrightarrow \exists$ reformulation

$$\sup \ c_2' x_2$$

$$s.t. \ x_1 \begin{pmatrix} \Lambda & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} + x_2 \begin{pmatrix} \times & \times & \times & M \\ \times & \Sigma & & \\ \times & & -I_s & \\ M^T & & & \end{pmatrix} \preceq \begin{pmatrix} I_p & & & \\ & I_{r-p} & & \\ & & & \\ & & & \end{pmatrix}$$

where $\Lambda \succ 0, \ M \neq 0, \ c_2' > 0, s \geq 0$.

Proof of $\Leftarrow M \neq 0 \Rightarrow x_2 = 0 \Rightarrow$ primal $= 0$.

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$
\begin{array}{ll}
\sup & c_2' x_2 \\
s.t. & x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array} \right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array} \right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \end{array} \right)
\end{array}
$$

where $\Lambda \succ 0,\ M \neq 0,\ c_2' > 0, s \geq 0.$

Proof of $\Leftarrow$ Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow \Lambda \bullet Y(1:p, 1:p) = 0$

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$\sup \; c_2' x_2$$

$$s.t. \;\; x_1 \begin{pmatrix} \Lambda & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} + x_2 \begin{pmatrix} \times & \times & \times & M \\ \times & \Sigma & & \\ \times & & -I_s & \\ M^T & & & \end{pmatrix} \preceq \begin{pmatrix} I_p & & & \\ & I_{r-p} & & \\ & & & \\ & & & \end{pmatrix}$$

where $\Lambda \succ 0, \; M \neq 0, \; c_2' > 0, s \geq 0.$

Proof of $\Leftarrow$ Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow \; \Lambda \bullet Y(1:p, 1:p) = 0$

$$\Rightarrow \; Y(1:p, 1:p) = 0$$

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$
\begin{array}{l}
\sup \; c_2' x_2 \\
s.t. \; x_1 \left(\begin{array}{c|c||c|c}
\Lambda & & & \\ \hline
 & & & \\ \hline\hline
 & & & \\ \hline
 & & &
\end{array}\right)
+ x_2 \left(\begin{array}{c|c||c|c}
\times & \times & \times & M \\ \hline
\times & \Sigma & & \\ \hline\hline
\times & & -I_s & \\ \hline
M^T & & &
\end{array}\right)
\preceq
\left(\begin{array}{c|c||c|c}
I_p & & & \\ \hline
 & I_{r-p} & & \\ \hline\hline
 & & & \\ \hline
 & & &
\end{array}\right)
\end{array}
$$

where $\Lambda \succ 0,\; M \neq 0,\; c_2' > 0, s \geq 0.$

Proof of $\Leftarrow$ Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow \; \Lambda \bullet Y(1:p, 1:p) = 0$

$\Rightarrow \; Y(1:p, 1:p) = 0$

$\Rightarrow$ 1st p rows and columns of Y are zero.

Suppose $m = 2$.
Then positive gap $\Leftrightarrow$ $\exists$ reformulation

$$\sup \; c_2' x_2$$

$$s.t. \; x_1 \begin{pmatrix} \Lambda & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} + x_2 \begin{pmatrix} \times & \times & \times & M \\ \times & \Sigma & & \\ \times & & -I_s & \\ M^T & & & \end{pmatrix} \preceq \begin{pmatrix} I_p & & & \\ & I_{r-p} & & \\ & & & \\ & & & \end{pmatrix}$$

where $\Lambda \succ 0, \; M \neq 0, \; c_2' > 0, s \geq 0.$

$\Rightarrow$ dual is

$$\inf \begin{pmatrix} I_{r-p} & 0 \\ 0 & 0 \end{pmatrix} \bullet Y'$$

$$s.t. \begin{pmatrix} \Sigma & 0 \\ 0 & -I_s \end{pmatrix} \bullet Y' = c_2' > 0$$

$$Y' \succeq 0,$$

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$\sup \; c_2' x_2$$

$$s.t. \; x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array} \right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array} \right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \\ \hline \end{array} \right)$$

where $\Lambda \succ 0, \; M \neq 0, \; c_2' > 0, s \geq 0.$

$\Rightarrow$ dual optimal value > 0.

Simple certificate of the positive pap

When does the underlying system admit a gap?

Given

$$(P_{SD}) \quad \sum_{i=1}^{m} x_i A_i \preceq B$$

is there $c \in \mathbb{R}^m$ such that there is a positive gap?

Suppose $m = 2$. Then $\exists (c_1, c_2)$ with positive gap
$\Leftrightarrow$ (P_{SD}) has reformulation

$$
x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array}\right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \\ \hline \end{array}\right),
$$

where $\Lambda \succ 0$, $M \neq 0$, $s \geq 0$.

How about $m > 2$?

Similar example with $m = 3$

$$
\begin{aligned}
\sup \quad & x_3 \\
s.t. \quad & x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.
\end{aligned}
$$

Similar example with $m = 3$

$$\sup \; x_3$$

$$s.t. \; x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

Primal $= 0$.

Similar example with $m = 3$

$$\sup \; x_3$$

$$s.t. \; x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

Primal $= 0.$

Dual : Variable $Y = (y_{ij}) \succeq 0$

Similar example with $m = 3$

$$\begin{array}{ll} \sup & x_3 \\ s.t. & x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}. \end{array}$$

Primal $= 0$.

Dual : Variable $Y = (y_{ij}) \succeq 0$

1st two dual constraints $\Rightarrow$ **1st two rows and columns of** Y **are zero.**

Similar example with $m = 3$

$$\begin{array}{ll} \sup & x_3 \\ s.t. & x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}. \end{array}$$

Primal $= 0.$

$\Rightarrow$ Dual is equivalent to:

$$\begin{array}{ll} \inf & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet Y' \\ s.t. & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet Y' = 1 \\ & Y' \succeq 0, \end{array}$$

Same structure as in the 2 variable case

Similar example with $m = 3$

$$\sup \ x_3$$

$$s.t. \ x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

Primal $= 0.$

$\Rightarrow$ Dual is equivalent to:

$$\begin{aligned} \inf \ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet Y' \\ s.t. \ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet Y' = 1 \\ & Y' \succeq 0, \end{aligned}$$

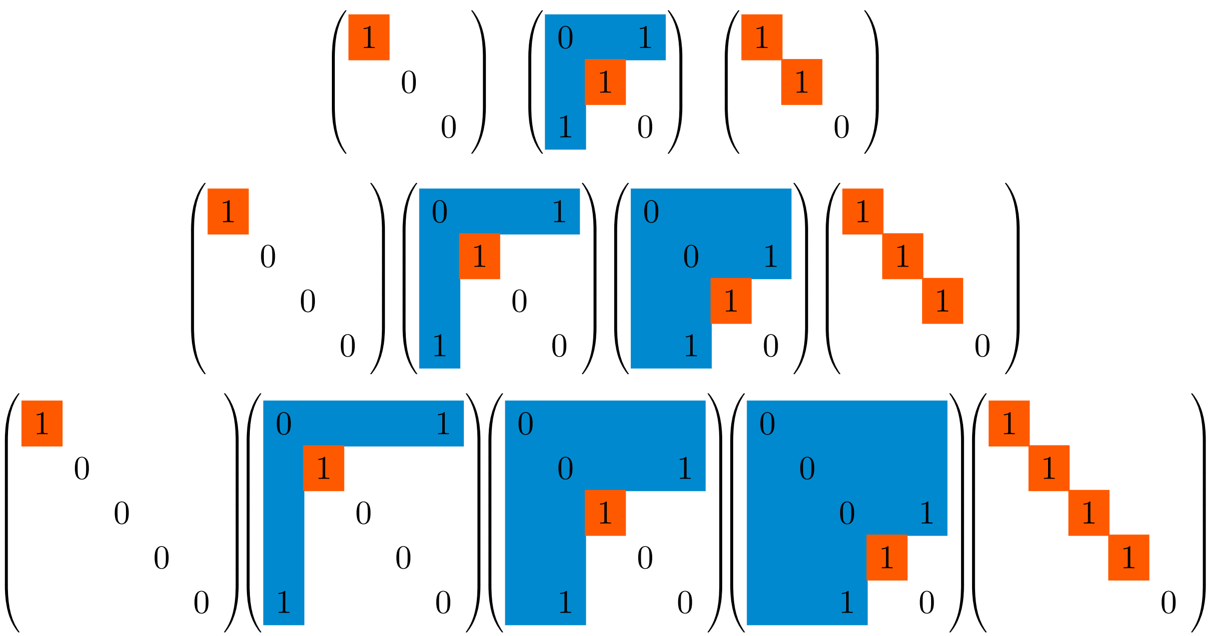
Same structure as in the 2 variable case

We can create such an instance for any $m = 2, 3, 4, \ldots$ with $n = m + 1$

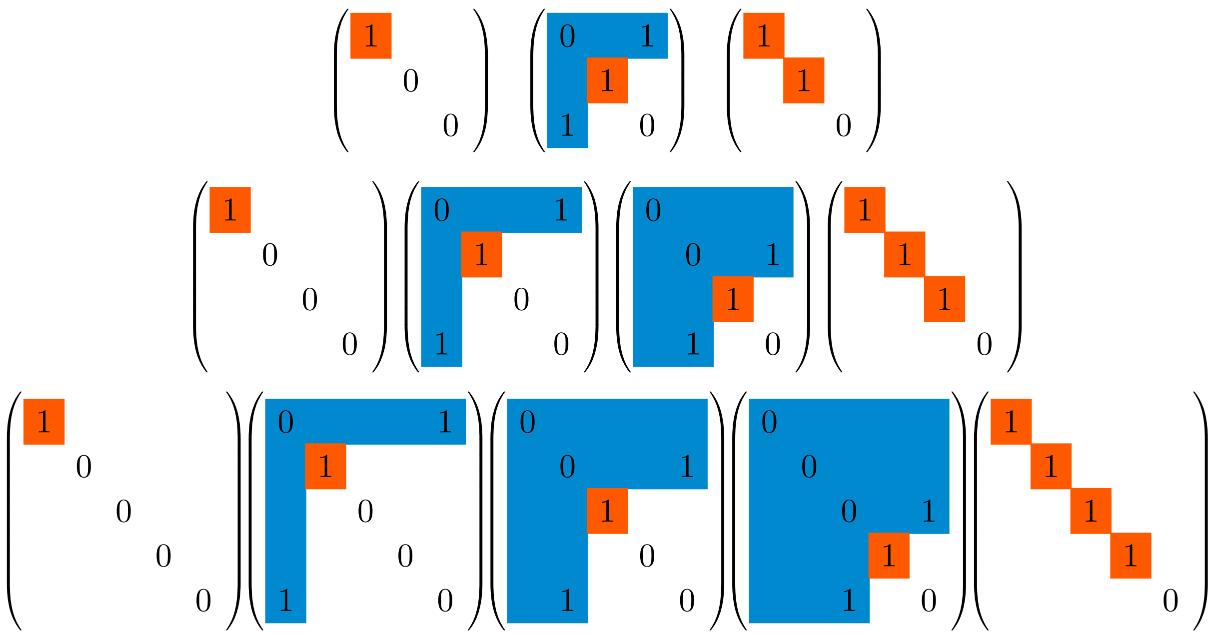
Name: **single sequence SDPs**

Objective is $\sup x_m$.

Sparsity structure when $m = 2, 3, 4$ ($n = m + 1$)



Sparsity structure when $m = 2, 3, 4$ ($n = m + 1$)



Blue stripe in A_{i+1} : If $Y \succeq 0,\ A_1 \bullet Y = \ldots A_i \bullet Y = 0,$ then these parts of Y are zero.

How do we get these instances? Background: facial reduction

Given H affine subspace, K closed convex cone s.t. $H \cap K \neq \emptyset$, a facial reduction algorithm (FRA) works as:

(1) If $\text{ri } K \cap H = \emptyset$, find $y \in H^\perp \cap (K^* \setminus K^\perp)$.

(2) Replace K by $K \cap y^\perp$. Goto (1).

How do we get these instances? Background: facial reduction

Given H affine subspace, K closed convex cone s.t. $H \cap K \neq \emptyset$, a facial reduction algorithm (FRA) works as:

(1) If ri $K \cap H = \emptyset$, find $y \in H^\perp \cap (K^* \setminus K^\perp)$.

(2) Replace K by $K \cap y^\perp$. Goto (1).

Facial reduction sequence:

The sequence $y_1, y_2, \ldots$ generated by the FRA.

How do we get these instances? Background: facial reduction

Given H affine subspace, K closed convex cone s.t. $H \cap K \neq \emptyset$, a facial reduction algorithm (FRA) works as:

(1) **If** $\mathrm{ri}\, K \cap H = \emptyset$, **find** $y \in H^{\perp} \cap (K^* \setminus K^{\perp})$.

(2) **Replace** K **by** $K \cap y^{\perp}$. **Goto (1).**

Facial reduction sequence:

The sequence $y_1, y_2, \ldots$ **generated by the FRA.**

Singularity degree:

Is the smallest number of FRA steps, until the FRA stops.

So we can talk about the singularity degree of an SDP.

Back to $m = 2$ example:

$$\begin{aligned} &\sup \; x_2 \\ &s.t. \; x_1 \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} + x_2 \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A_2} \preceq \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{B} \end{aligned}$$

Here (A_1) is a facial reduction sequence for (D).

Back to $m = 2$ example:

$$\sup \; x_2$$

$$\text{s.t.} \; x_1 \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} + x_2 \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A_2} \preceq \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{B}$$

Here (A_1) is a facial reduction sequence for (D).

$A_1 \bullet Y = 0$ proves that dual matrix must look like

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{23} & y_{33} \end{pmatrix}$$

Back to larger example:

$$
\begin{aligned}
&\sup \; x_3 \\
&s.t. \; x_1 \underbrace{\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}}_{A_1} + x_2 \underbrace{\begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix}}_{A_2} + x_3 \underbrace{\begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix}}_{A_3} \preceq \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}}_{B}
\end{aligned}
$$

Here (A_1, A_2) is a facial reduction sequence for (D).

Back to larger example:

$$\sup \; x_3$$

$$s.t. \; x_1 \underbrace{\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}}_{A_1} + x_2 \underbrace{\begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix}}_{A_2} + x_3 \underbrace{\begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix}}_{A_3} \preceq \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}}_{B}$$

Here (A_1, A_2) is a facial reduction sequence for (D).

$A_1 \bullet Y = A_2 \bullet Y = 0$ proves that dual matrix must look like

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y_{33} & y_{34} \\ 0 & 0 & y_{43} & y_{44} \end{pmatrix}$$

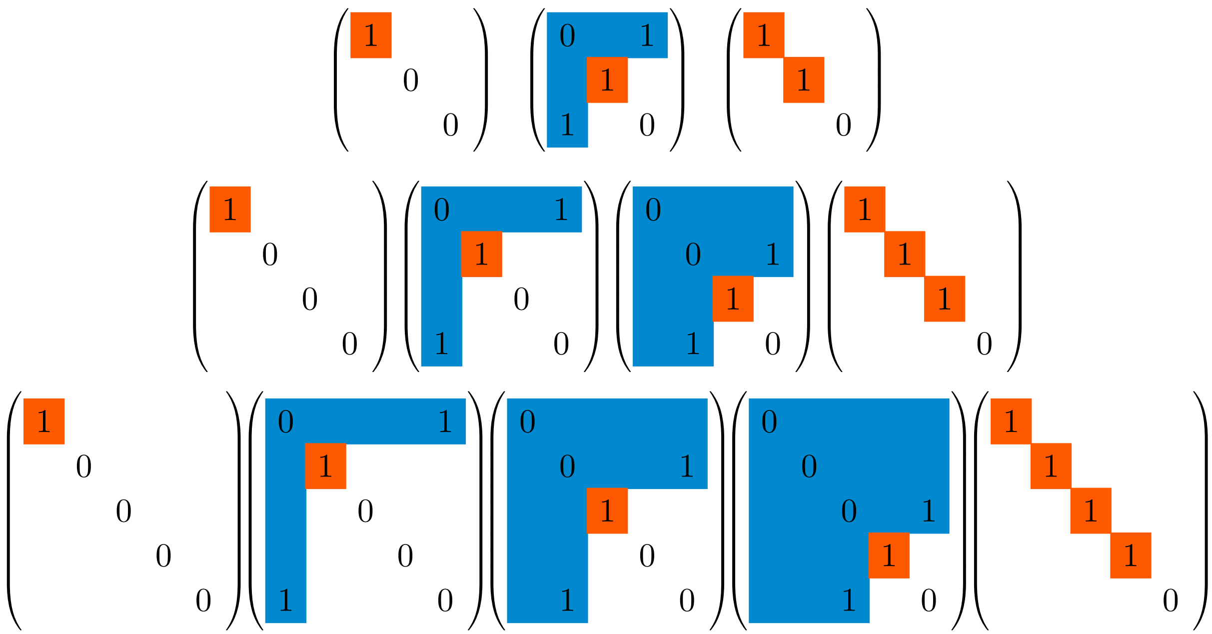
Theorem:

- $\mathrm{sing(D)} \leq m$.
- $= m \Rightarrow$ no gap. (Easy)
- $\forall m \geq 2$, $\forall g > 0$ $\exists$ instance s.t.

$$\mathrm{sing(D)} = m - 1 \text{ and gap is } g.$$

Indeed, the single sequence SDPs have

$\mathrm{sing}(\mathrm{D}) = m - 1$



We also look at the homogeneous dual

$$\begin{aligned} A_i \bullet Y &= 0 \, \forall i \\ B \bullet Y &= 0 \quad (HD) \\ Y &\succeq 0 \end{aligned}$$

Theorem:

- $\mathrm{sing(HD)} \leq m + 1$. (Easy)
- $= m + 1 \Rightarrow$ no gap.
- $\forall m \geq 2,\ \forall g > 0\ \exists$ instance s.t.

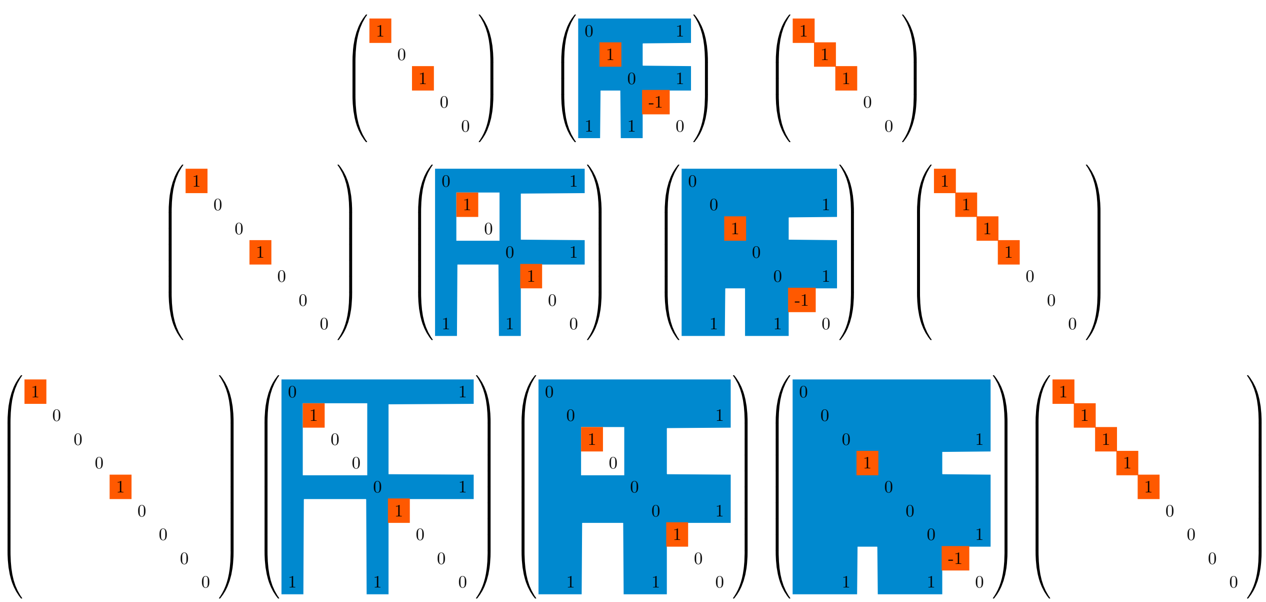
  $\mathrm{sing(D)} = m - 1$ and $\mathrm{sing(HD)} = m$ and gap is g.

Theorem:

- $\mathbf{sing(HD) \le m+1.}$ **(Easy)**
- $\mathbf{= m+1 \Rightarrow}$ **no gap.**
- $\mathbf{\forall m \ge 2,\ \forall g > 0\ \exists}$ **instance s.t.**

  $\mathbf{sing(D) = m-1}$ **and** $\mathbf{sing(HD) = m}$ **and gap is** $\mathbf{g.}$

Proof: the double sequence SDPs



Surprising connection between singularity degrees

$$\underbrace{\text{sing(HD)} = m+1}_{\text{(maximal)}} \;\Rightarrow\; \underbrace{\text{sing(D)} = 0}_{\text{(minimal)}}$$

sing(D) = 0 means it satisfies Slater's condition

Problem library and computational results

- **Instances with $m = 2, 3, \ldots, 11$; gap = 10.**

- **Integral data. Gap can be verified by hand, in exact arithmetic.**

Problem library and computational results

- **Instances with $m = 2, 3, \ldots, 11$; gap = 10.**

- **Integral data. Gap can be verified by hand, in exact arithmetic.**

- **Four categories:**
  - $gap\_single\_finite\_clean\_m$
  - $gap\_single\_finite\_messy\_m$
  - $gap\_single\_inf\_clean\_m$,
  - $gap\_single\_inf\_messy\_m$.

- **Messy means we applied a similarity transformation $T^T()T$.**

Results

	GAP, SINGLE, FINITE		GAP, SINGLE, INFINITE	
	CLEAN	MESSY	CLEAN	MESSY
MOSEK	1	1	0	0
SDPA-GMP	1	1	0	0
PP+MOSEK	10	1	10	0
SIEVE-SDP + MOSEK	10	1	10	0

- PP: preprocessor of Permenter and Parrilo
- Sieve-SDP: preprocessor of Zhu, Pataki, Tran-Dinh

Note: SPECTRA works when $m = 2$.

Summary:

- **Positive gaps: "worst/most interesting" pathology of SDPs.**
- **Complete characterization for $m = 2$ by reformulation.**
- **Complete characterization of positive gap systems with $m = 2$.**

Summary:

- **Positive gaps: "worst/most interesting" pathology of SDPs.**
- **Complete characterization for $m = 2$ by reformulation.**
- **Complete characterization of positive gap systems with $m = 2$.**
- **Similarly structured positive gap SDPs in any dimension.**
- **Highest singularity degree that leads to a positive gap.**
- **Challenging problem library.**

Papers

- **P:** Bad semidefinite programs: they all look the same, **2010–SIOPT 2017**
- **P:** Characterizing bad semidefinite programs: normal forms and short proofs **SIAM Review, 2019**
- **P:** On positive duality gaps in semidefinite programming, submitted.

Thank you!