Combinatorial Characterizations in Semidefinite Programming Duality: how Elementary Row Operations Help

Gábor Pataki

Department of Statistics and Operations Research

UNC Chapel Hill

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A Semidefinite Program (SDP)

$$\begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B. \end{array} \qquad (SDP)$$

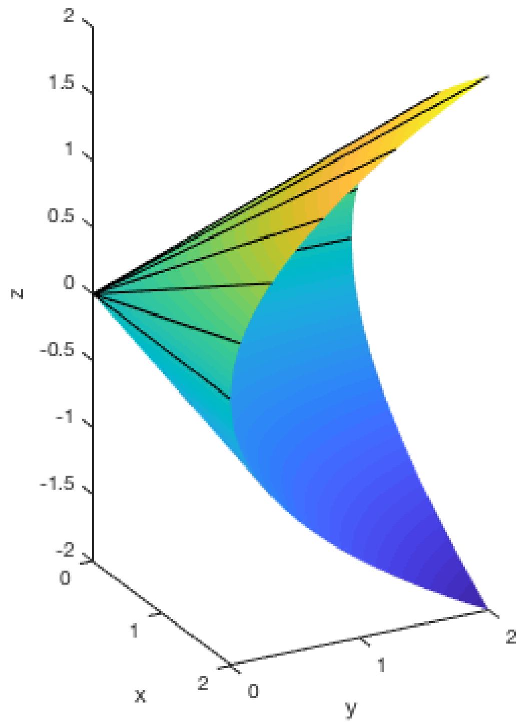
Here

- A_i, B are symmetric matrices, $c, x \in \mathbb{R}^m$.
- $A \preceq B$ means that $B - A$ is symmetric positive semidefinite (psd).
- An $n \times n$ matrix Y is positive semidefinite, if all principal subdeterminants are nonnegative.
- Equivalently, if $v^T Y v \geq 0 \, \forall v \in \mathbb{R}^n$.

Some nice pictures of SDP feasible sets, 1

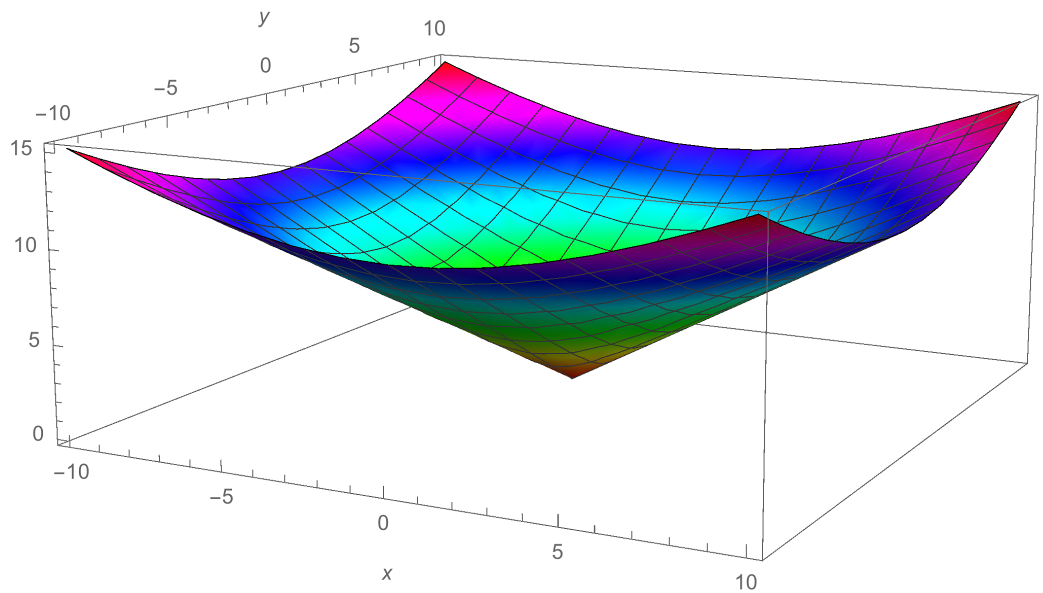
Picture of $\mathbf{2 \times 2}$ psd cone:

$$\left\{ (x, y, z) : \begin{pmatrix} x & z \\ z & y \end{pmatrix} \succeq 0 \right\}$$



Some nice pictures of SDP feasible sets, 2

$$\left\{ (x, y, z) : zI \succeq \begin{pmatrix} x+1 & -y \\ -y & -x+1 \end{pmatrix} \right\}$$



Why is SDP important:
LP $\subseteq$ SDP $\subseteq$ Convex Optimization

LP (Linear Program) as SDP:

- If A_i and B are diagonal $\Rightarrow$ so is $B - \sum_{i=1}^{m} x_i A_i$.
- So it is psd iff diagonal elements are nonnegative.
- So LP can be modeled as SDP.

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- So LP can be modeled as SDP.

SDP is a convex problem:

- Feasible set is convex, since set of psd matrices is.

Why is SDP important: applications in

- 0–1 Integer programming.
- Approx algorithms
- Chemical engineering
- Chemistry
- Coding theory
- Control theory
- Combinatorial opt
- Discrete geometry
- Eigenvalue optimization
- Facility planning
- Finance
- Geometric optimization
- Global optimization
- Graph visualization
- Inventory theory
- Machine learning
- Matrix analysis
- PDEs
- Probability theory
- Robust optimization
- Signal processing
- Statistics
- Structural optimization

Why is SDP important: beautiful theory in

- Duality
- Interior point methods
- Geometry

. . . and many interesting glitches

- **Interior point methods do not work as well as in LP.**
- **Nor does duality theory.**

SDP in a different shape

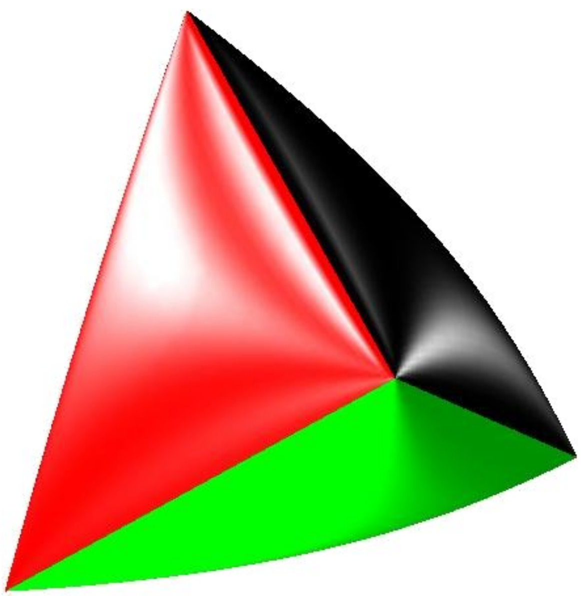
$$\begin{aligned} \inf_Y \;& B \bullet Y \\ s.t. \;& Y \succeq 0 \\ & A_i \bullet Y = c_i \, (i = 1, \ldots, m). \end{aligned}$$

Here

- A_i, B are symmetric matrices, $c \in \mathbb{R}^m$.
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$
- Example: $\{Y \succeq 0 \,|\, y_{ii} = 1\}$ the set of correlation matrices.

3 by 3 correlation matrices

The set $\left\{ (x, y, z) \mid \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$



SDP duality

The primal-dual pair of SDPs:

$$\sup_x \; c^T x \qquad\qquad \inf_Y \; B \bullet Y$$

$$s.t. \; \sum_{i=1}^m x_i A_i \preceq B \qquad\qquad Y \succeq 0$$

$$A_i \bullet Y = c_i \; (i = 1, \ldots, m).$$

Easy: If x, Y are feasible $\Rightarrow \; c^T x \leq B \bullet Y$.

Ideally: $\exists x^*, Y^*$ such that $c^T x^* = B \bullet Y^*$.

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But: SDPs, unlike LPs can be **pathological:** nonattainment, positive gaps.

Pathological SDPs often defeat SDP solvers.

Ben-Tal, Nemirovsky: "Is there something wrong with SDP duality?"

Pathology # 1: nonattainment in dual

Primal:

$$\begin{array}{ll} \sup & 2x_1 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \sup & 2x_1 \\ s.t. & \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0 \end{array}$$

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Unattained $\inf = 0 :$ $y_{11} > 0$ **is feasible, but** $y_{11} = 0$ **is not.**

Same story in pictures

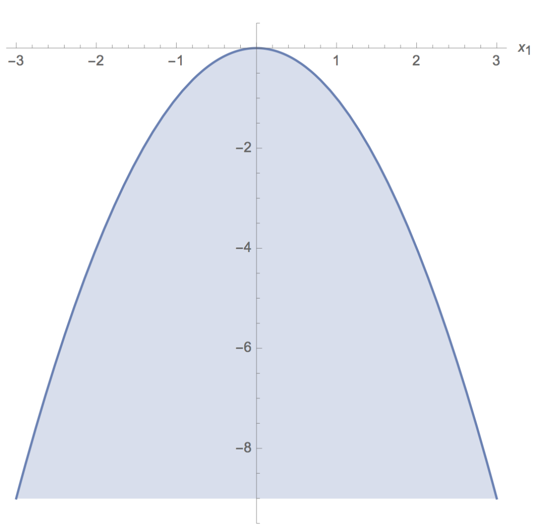
Primal:

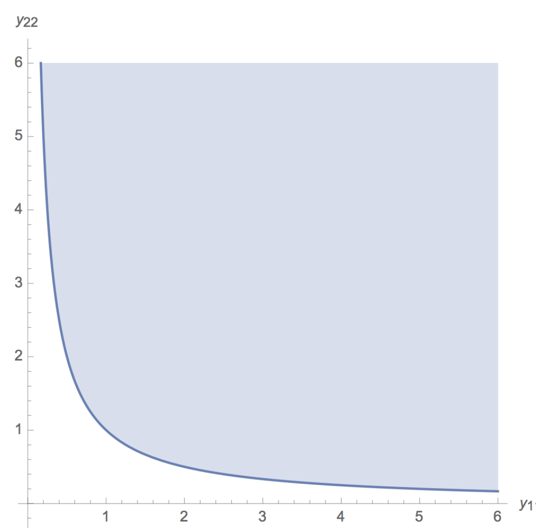
$$\sup \ 2x_1$$
$$s.t. \ -x_1^2 \geq 0$$

Dual:

$$\inf \ y_{11}$$
$$s.t. \ y_{11}y_{22} \geq 1$$
$$y_{11} \geq 0, \ y_{22} \geq 0.$$

Highest point on degenerate parabola vs. leftmost point on hyperbola





Pathology # 2: positive duality gap

Primal:

$$\sup \; x_2$$

$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Dual value is 1, and it is attained.

Bad behavior defined

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$$(P_{SD}) \ \sum_{i=1}^{m} x_i A_i \preceq B$$

- We say that it is badly behaved if $\exists c$ such that

  $$\sup\{ c^T x \mid x \in (P_{SD}) \} < +\infty$$

  but the dual program has no solution with same value (i.e. dual does not attain, or positive gap).
- Well behaved, otherwise.
- A slack is $Z = B - \sum_i x_i A_i \succeq 0$.

Motivation

The systems

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Curious similarity – of these, and about 20 others in the literature

Why all bad SDPs look the same

- **Semidefinite system:**

$$(P_{SD}) \quad \sum_{i=1}^{m} x_i A_i \preceq B$$

- **W.l.o.g. the max (rank) slack is**

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- Then (P_{SD}) badly behaved $\Leftrightarrow$ $\exists V$ a lin. combination of the A_i as

$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0,\ \mathrm{R}(V_{12}^T) \not\subseteq \mathrm{R}(V_{22}).$$

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What is missing?

- Matrices Z, V prove that (P_{SD}) is badly behaved.
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Reformulations of

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are obtained by a sequence of:

- Apply a rotation $V^T()V$ to all matrices, where V is invertible.
- $B \leftarrow B + \sum_{i=1}^{m} \mu_i A_i$
- $A_i \leftarrow \sum_{j=1}^{m} \lambda_j A_j$ where $\lambda_i \neq 0$
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Mostly: just elementary row operations done on (D). E.g. exchange constraints

$$A_i \bullet Y = c_i \;\; \text{and} \;\; A_j \bullet Y = c_j$$

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad}) \quad \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

where

1) Z is max slack; 2) $\begin{pmatrix} G_i \\ H_i \end{pmatrix}$ lin. indep. 3) $H_m \succeq 0$

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How to get there? Block Gaussian elimination!

$$\begin{pmatrix} \text{vec } A_1 \\ \vdots \\ \text{vec } A_m \end{pmatrix} \rightarrow \begin{pmatrix} \overbrace{\times}^{F_i} & \overbrace{\quad 0 \quad}^{(G_i, H_i)} \\ \times & \times \end{pmatrix}$$

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Proof that $(P_{SD,bad})$ is badly behaved:

$$\begin{aligned} x \text{ feasible } &\Rightarrow x_{k+1} = \cdots = x_m = 0 \\ &\Rightarrow \quad \sup -x_m = 0 \end{aligned}$$

But: no dual soln with value 0

Example: before and after

$$x_1 \begin{pmatrix} 6 & 10 \\ 10 & 16 \end{pmatrix} \preceq \begin{pmatrix} 19 & 32 \\ 32 & 52 \end{pmatrix}$$

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After reformulation:

$$x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which is trivially badly behaved.

Large example: before reformulation

$$x_1 \begin{pmatrix} 54 & 46 & 50 & 4 \\ 46 & -38 & 87 & -106 \\ 50 & 87 & -60 & 296 \\ 4 & -106 & 296 & -368 \end{pmatrix} + x_2 \begin{pmatrix} 110 & 91 & 105 & -6 \\ 91 & -72 & 171 & -210 \\ 105 & 171 & -72 & 528 \\ -6 & -210 & 528 & -672 \end{pmatrix} + x_3 \begin{pmatrix} 42 & 35 & 40 & 0 \\ 35 & -28 & 67 & -8 \\ 40 & 67 & -36 & 21 \\ 0 & -82 & 216 & -27 \end{pmatrix}$$

$$+ x_4 \begin{pmatrix} 36 & 30 & 35 & -2 \\ 30 & -24 & 57 & -70 \\ 35 & 57 & -24 & 176 \\ -2 & -70 & 176 & -224 \end{pmatrix} \preceq \begin{pmatrix} 389 & 323 & 370 & -12 \\ 323 & -257 & 610 & -74 \\ 370 & 610 & -288 & 192 \\ -12 & -748 & 1920 & -243 \end{pmatrix}$$

Hard to tell if well or badly behaved

Large example: after reformulation

$$x_1\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + x_2\left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + x_3\left(\begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & -1 \\ \hline 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{array}\right)$$

$$+x_4\left(\begin{array}{cc|cc} 0 & 0 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ \hline 3 & 2 & 4 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \preceq \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

As before: $x_3 = x_4 = 0 \Rightarrow \sup -x_4 = 0$

But: no dual solution with value 0

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How about proving infeasibility?

This part is joint with Minghui Liu.

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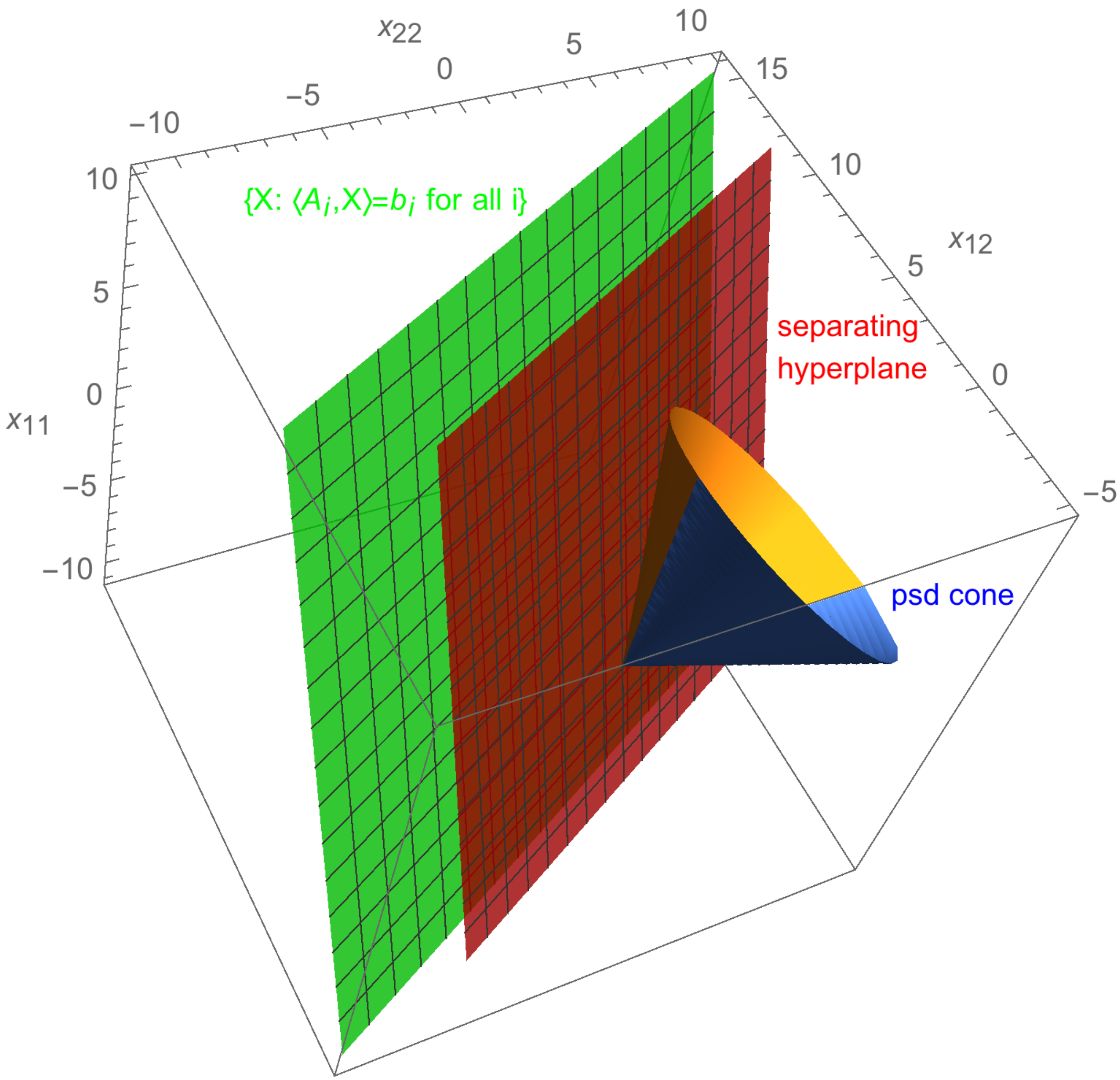
- Thorny issue: How to prove infeasibility?

Preferable way: by a separating hyperplane



Such a hyperplane may not exist!

On one hand

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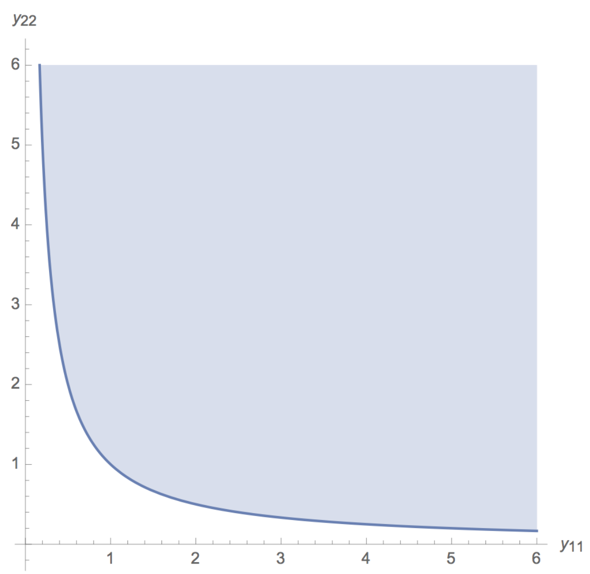
That hyperbola again . . .



Literature: exact certificates of infeasibility

- Ramana 1995
- Ramana, Tuncel, Wolkowicz, 1997
- Klep, Schweighofer 2013
- Waki, Muramatsu 2013: variant of facial reduction of
- Borwein, Wolkowicz 1981
- These are more involved than a separating hyperplane

Ideas from Gaussian elimination

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- **We will borrow ideas from the row echelon form.**

Infeasible example, and proof of infeasibility

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet X = -1$$

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- **Suppose X feasible** $\Rightarrow X_{11} = 0$
  $\Rightarrow X_{12} = X_{13} = 0$
  $\Rightarrow X_{22} = -1$
- **Main idea: We will find such a structure in every infeasible semidefinite system.**

Reformulation

$$\begin{aligned} A_i \bullet X &= b_i \ (i = 1, \ldots, m) \\ X &\succeq 0 \end{aligned} \qquad (\mathrm{P})$$

- We obtain a reformulation of (P) by a sequence of the following:

(1) Elementery row operations on the equations.
(2) $A_i \leftarrow V^T A_i V \ (i = 1, , \ldots, m)$, where V is invertible.

- (1) is inherited from Gaussian elimination.

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- (1) is inherited from Gaussian elimination.

- **Fact:** Reformulations preserve (in)feasibility.

Some linear algebra

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \bullet X = 0, \; X \succeq 0 \Rightarrow ?$$

Some linear algebra

$$\begin{pmatrix} \boldsymbol{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bullet \boldsymbol{X} = 0,\ \boldsymbol{X} \succeq \mathbf{0} \Rightarrow \boldsymbol{X} = \begin{pmatrix} \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \boldsymbol{X}_{22} \end{pmatrix},\ \boldsymbol{X}_{22} \succeq \mathbf{0}.$$

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_{k+1}' \bullet X &= -1 \qquad\qquad (\mathrm{P_{ref}}) \\ &\vdots \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A_i' look like

$$A_1' = \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix}, \ A_i' = \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

with $r_1, \ldots, r_{k+1} \geq 0$.

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$\Rightarrow A_{k+1}' \bullet X \geq 0$

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$$\begin{matrix} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{A'_1} & \bullet X = 0 \\ \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A'_2} & \bullet X = -1 \end{matrix}$$

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Proof outline

- **Based on simplified facial reduction algorithm: construct the A_i' one by one.**
- **"Difficult" direction is about 1.5 pages.**
- **Alternative: adapt a traditional facial reduction algorithm, the closest one is by Waki and Muramatsu.**

Is this just theory?

- **We cannot construct the reformulations in poly time :(**
- **To do so, we would need to solve SDPs exactly.**
- **However...**

Application 1: simple proof that SDP feasibility is in $NP \cap coNP$ in real number model

- **Proof of NP: show feasible X.**

Application 1: simple proof that SDP feasibility is in $NP \cap coNP$ in real number model

- **Proof of NP: show feasible X.**
- **Proof of co-NP: reformulation and how we got it:**
  - $V \in \mathbb{R}^{n \times n}$ **to encode all similarity transformations.**
  - $T \in \mathbb{R}^{m \times m}$ **to encode elementary row ops.**

Application 2: generating infeasible SDPs

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with $r_1, \dots, r_{k+1} \geq 0$.

- Using this result, we can generate **all** infeasible SDP problems, as:

(1) Generate a system like $(\mathrm{P_{ref}})$.

(2) Reformulate it.

Application 2: generating infeasible SDPs

- **We can generate challenging instances!**

- **Problem library by Liu-P 2016: infeasible and weakly infeasible SDPs**

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- **We can generate challenging instances!**
- **Problem library by Liu-P 2016: infeasible and weakly infeasible SDPs**
- **As to solving them: Douglas-Rachford splitting of Liu–Ryu–Yin 2017;**
- **Homotopy method of Hauenstein,Liddell, Zhang 2018**

Application 3: recognizing infeasibility in practice

- **Sometimes we do not even have to reformulate an SDP to find the trivial structure that proves infeasibilty ...or to reduce the SDP.**

- **Zhu–P–Tran-Dinh Sieve-SDP preprocessor**

Application 3: recognizing infeasibility in practice

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- **Zhu–P–Tran-Dinh Sieve-SDP preprocessor**
- **Before and after picture of an SDP**

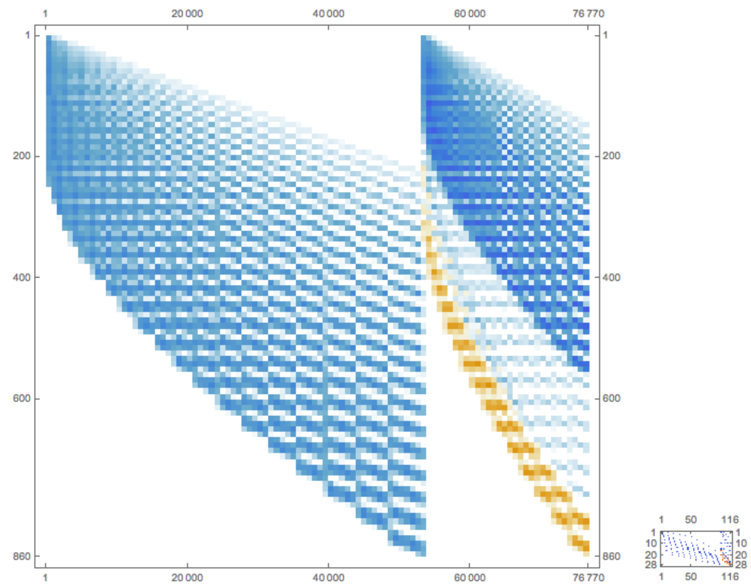

Figure 1: Instance “ex4.2_order20”: size and sparsity before and after preprocessing

Conclusion

- **Pathologies in duality:** **well- and badly behaved semidefinite (feasible) systems.**

- **Combinatorial type characterizations.**

- **Reformulations** **to easily recognize good and bad behavior** $\rightarrow NP \cap co - NP$ **certificates.**

Conclusion

- **Pathologies in duality: well- and badly behaved semidefinite (feasible) systems.**
- **Combinatorial type characterizations.**
- **Reformulations to easily recognize good and bad behavior $\rightarrow NP \cap co - NP$ certificates.**
- **Exact, simple certificate of infeasibility of a semidefinite system based on elementary reformulation.**
- **Algorithm to systematically generate all infeasible SDPs.**
- **Other application: preprocessing by Sieve-SDP.**

Papers

- **P:** **Bad semidefinite programs: they all look the same, 2010–SIOPT 2017**
- **Liu–P:** **Exact duality in semidefinite programming based on elementary reformulations, SIOPT 2015**
- **Liu–P:** **Exact duals and short certificates fo infeasibility and weak infeasibility in conic linear programming, Math. Programming 2017.**

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- **Liu–P:** Exact duals and short certificates fo infeasibility and weak infeasibility in conic linear programming, **Math. Programming 2017**.
- **P:** Characterizing bad semidefinite programs: normal forms and short proofs
  SIAM Review, to appear
- **Zhu–P–Tran-Dinh:** Sieve-SDP: a simple algorithm to pre-process semidefinite programs
  Mathematical Programming Computation, 2019
- **P:** On positive duality gaps in semidefinite programming, **2018**

Thank you!