Combinatorial Characterizations in Semidefinite Programming Duality: how Elementary Row Operations Help

Gábor Pataki

Department of Statistics and Operations Research

UNC Chapel Hill

Talk at the Simons Institute, April 2019

A pair of Semidefinite Programs (SDP)

$$\begin{array}{llll} \sup_x & c^T x & \inf_Y & B \bullet Y \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B & & Y \succeq 0 \\ & & & A_i \bullet Y = c_i \, (i = 1, \ldots, m). \end{array}$$

Here

- A_i, B are symmetric matrices, $c, x \in \mathbb{R}^m$.
- $A \preceq B$ means: $B - A$ is positive semidefinite (psd).
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

A pair of Semidefinite Programs (SDP)

$$\begin{array}{llll} \sup_x & c^T x & \inf_Y & B \bullet Y \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B & & Y \succeq 0 \\ & & & A_i \bullet Y = c_i \, (i = 1, \ldots, m). \end{array}$$

Here

- A_i, B are symmetric matrices, $c, x \in \mathbb{R}^m$.
- $A \preceq B$ means: $B - A$ is positive semidefinite (psd).
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.
- $Y \succeq 0 \overset{\text{def}}{\Leftrightarrow}$ all principal subdeterminants are nonnegative.
- Equivalently, if $v^T Y v \geq 0 \, \forall v \in \mathbb{R}^n$.

Why is SDP important: LP $\subseteq$ SDP $\subseteq$ Convex Optimization

LP as SDP:

- If A_i and B are diagonal $\Rightarrow$ so is $B - \sum_{i=1}^{m} x_i A_i$.
- So it is psd iff diagonal elements are nonnegative.
- So LP can be modeled as SDP: make A_i, B diagonal.

Why is SDP important:
LP $\subseteq$ SDP $\subseteq$ Convex Optimization

LP as SDP:

- If A_i and B are diagonal $\Rightarrow$ so is $B - \sum_{i=1}^{m} x_i A_i$.
- So it is psd iff diagonal elements are nonnegative.
- So LP can be modeled as SDP: make A_i, B diagonal.

SDP is a convex problem:

- Feasible set is convex, since set of psd matrices is.

SDP duality

The primal-dual pair of SDPs:

$$\begin{array}{ll} \sup_x \ c^T x & \inf_Y \ B \bullet Y \\ \quad s.t. \ \sum_{i=1}^m x_i A_i \preceq B & \quad Y \succeq 0 \\ & \quad A_i \bullet Y = c_i \ (i = 1, \ldots, m). \end{array}$$

SDP duality

The primal-dual pair of SDPs:

$$\begin{array}{ll} \sup_x \ c^T x & \inf_Y \ B \bullet Y \\ \quad s.t. \ \sum_{i=1}^m x_i A_i \preceq B & \quad Y \succeq 0 \\ & \quad A_i \bullet Y = c_i \, (i = 1, \ldots, m). \end{array}$$

Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

SDP duality

The primal-dual pair of SDPs:

$$\begin{array}{llll} \sup_x & c^T x & \quad \inf_Y & B \bullet Y \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B & & Y \succeq 0 \\ & & & A_i \bullet Y = c_i \; (i = 1, \ldots, m). \end{array}$$

Easy: If x and Y are feasible, then $c^T x \le B \bullet Y$.

Ideal situation: $\exists \bar{x}, \exists \bar{Y} : c^T \bar{x} = B \bullet \bar{Y}$.

SDP duality

The primal-dual pair of SDPs:

$$
\begin{array}{llll}
\sup_x & c^T x & \inf_Y & B \bullet Y \\
s.t. & \sum_{i=1}^m x_i A_i \preceq B & & Y \succeq 0 \\
& & & A_i \bullet Y = c_i \, (i = 1, \ldots, m).
\end{array}
$$

Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

Ideal situation: $\exists \bar{x}, \exists \bar{Y} : c^T \bar{x} = B \bullet \bar{Y}$.

But: in SDP, unlike in LP pathological phenomena occur: nonattainment, positive gaps.

This is bad, since we would like a certificate of optimality.

SDP duality

The primal-dual pair of SDPs:

$$\begin{array}{llll} \sup_x & c^T x & \inf_Y & B \bullet Y \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B & & Y \succeq 0 \\ & & & A_i \bullet Y = c_i \, (i = 1, \dots, m). \end{array}$$

Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

Ideal situation: $\exists \bar{x}, \exists \bar{Y} : c^T \bar{x} = B \bullet \bar{Y}$.

But: in SDP, unlike in LP pathological phenomena occur: nonattainment, positive gaps.

This is bad, since we would like a certificate of optimality.

Pathological SDPs often defeat SDP solvers.

Pathology # 1: nonattainment in dual

Primal:

$$\begin{array}{ll} \sup & 2x_1 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \sup & 2x_1 \\ s.t. & \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0 \end{array}$$

Pathology # 1: nonattainment in dual

Primal:

$$\begin{array}{ll} \sup & 2x_1 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \sup & 2x_1 \\ s.t. & \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0 \end{array}$$

Only feasible x_1 **is** $x_1 = 0.$

Pathology # 1: nonattainment in dual

Primal:

$$\begin{array}{ll} \sup & 2x_1 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \sup & 2x_1 \\ s.t. & \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0 \end{array}$$

Only feasible x_1 **is** $x_1 = 0$.

Dual: Dual variable is $Y \succeq 0$.

$$\begin{array}{ll} \inf & y_{11} \\ s.t. & \begin{pmatrix} y_{11} & 1 \\ 1 & y_{22} \end{pmatrix} \succeq 0 \end{array}$$

Pathology # 1: nonattainment in dual

Primal:

$$\begin{array}{ll} \sup & 2x_1 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \sup & 2x_1 \\ s.t. & \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0 \end{array}$$

Only feasible x_1 **is** $x_1 = 0.$

Dual: Dual variable is $Y \succeq 0.$

$$\begin{array}{ll} \inf & y_{11} \\ s.t. & \begin{pmatrix} y_{11} & 1 \\ 1 & y_{22} \end{pmatrix} \succeq 0 \end{array}$$

Unattained $\inf = 0:$ $y_{11} > 0$ **is feasible,** but $y_{11} = 0$ **is not.**

Pathology # 2: positive duality gap

Primal:

$$\begin{array}{ll} \sup & x_2 \\ s.t. & x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

Pathology # 2: positive duality gap

Primal:

$$\sup \; x_2$$
$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Only feasible x_2 **is** $x_2 = 0.$

Pathology # 2: positive duality gap

Primal:

$$\sup \; x_2$$

$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Only feasible x_2 is $x_2 = 0$.

Dual value is 1, and it is attained.

Terminology

Definition:

- **The system**

$$(P_{SD}) \ \sum_{i=1}^{m} x_i A_i \preceq B$$

is badly behaved if $\exists c$ such that

$$\sup\{ c^T x \mid x \in (P_{SD}) \} < +\infty$$

but the dual program has no solution with same value (i.e. dual does not attain, or positive gap).

Terminology

Definition:

- **The system**

$$(P_{SD})\ \sum_{i=1}^m x_i A_i \preceq B$$

is badly behaved if $\exists c$ such that

$$\sup\{\, c^T x \mid x \in (P_{SD})\,\} < +\infty$$

but the dual program has no solution with same value (i.e. dual does not attain, or positive gap).

- **Well behaved, otherwise.**
- **A slack is $Z = B - \sum_i x_i A_i \succeq 0$**

Motivation

The systems

$$x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are both badly behaved.

Motivation

The systems

$$x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are both badly behaved.

Curious similarity – of these, and about 20 others in the literature

Why all bad SDPs look the same

- **Semidefinite system:**

$$(P_{SD}) \quad \sum_{i=1}^{m} x_i A_i \preceq B$$

Why all bad SDPs look the same

- Semidefinite system:

$$(P_{SD}) \quad \sum_{i=1}^m x_i A_i \preceq B$$

- W.l.o.g. the max (rank) slack is

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (P_{SD}) badly behaved $\Leftrightarrow$ $\exists V$ a lin. combination of the A_i as

$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0,\ \mathrm{R}(V_{12}^T) \not\subseteq \mathrm{R}(V_{22}).$$

Why all bad SDPs look the same

- Semidefinite system:

$$(P_{SD}) \quad \sum_{i=1}^{m} x_i A_i \preceq B$$

- W.l.o.g. the max (rank) slack is

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (P_{SD}) badly behaved $\Leftrightarrow$ $\exists V$ a lin. combination of the A_i as

$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0,\ \mathrm{R}(V_{12}^T) \not\subseteq \mathrm{R}(V_{22}).$$

- Ex: $x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Why all bad SDPs look the same

- **Semidefinite system:**

$$(P_{SD}) \quad \sum_{i=1}^m x_i A_i \preceq B$$

- **W.l.o.g. the max (rank) slack is**

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (P_{SD}) **badly behaved** $\Leftrightarrow$ $\exists V$ **a lin. combination of the** A_i **as**

$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0, \ \mathrm{R}(V_{12}^T) \not\subseteq \mathrm{R}(V_{22}).$$

- **Ex:** $x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^{Z}$

Why all bad SDPs look the same

- Semidefinite system:

$$(P_{SD}) \quad \sum_{i=1}^{m} x_i A_i \preceq B$$

- W.l.o.g. the max (rank) slack is

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (P_{SD}) badly behaved $\Leftrightarrow$ $\exists V$ a lin. combination of the A_i as

$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0, \ \mathrm{R}(V_{12}^T) \not\subseteq \mathrm{R}(V_{22}).$$

- Ex: $x_1 \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{V} \preceq \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^{Z}$

What is missing?

- Matrices Z, V prove that (P_{SD}) is badly behaved.
- But: they do not provide a "bad" c objective function
- Nor a poly time, or easy to verify proof of bad behavior

What is missing?

- Matrices Z, V prove that (P_{SD}) is badly behaved.
- But: they do not provide a “bad” c objective function
- Nor a poly time, or easy to verify proof of bad behavior
- Aside: how do we prove that $Ax = b$ is infeasible?

What is missing?

- Matrices Z, V prove that (P_{SD}) is badly behaved.
- But: they do not provide a "bad" c objective function
- Nor a poly time, or easy to verify proof of bad behavior
- Aside: how do we prove that $Ax = b$ is infeasible? $\rightarrow$ row echelon form.

What is missing?

- **Matrices Z, V prove that (P_{SD}) is badly behaved.**
- **But: they do not provide a "bad" c objective function**
- **Nor a poly time, or easy to verify proof of bad behavior**
- **Aside: how do we prove that $Ax = b$ is infeasible? $\rightarrow$ row echelon form.**
- **We will borrow ideas from the row echelon form.**

Reformulations of

$$(P_{SD})\ \sum_{i=1}^{m} x_i A_i \preceq B$$

are obtained by a sequence of:

- Apply a rotation $V^T()V$ to all matrices, where V is invertible.
- $B \leftarrow B + \sum_{i=1}^{m} \mu_i A_i$
- $A_i \leftarrow \sum_{j=1}^{m} \lambda_j A_j$ where $\lambda_i \neq 0$
- Exchange A_i and A_j.

Reformulations preserve well/badly behaved status; preserve max slack; provide an equivalence relation

Theorem: (P_{SD}) **is badly behaved** $\Leftrightarrow$ **it has a reformulation:**

$$(P_{SD,bad}) \ \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad})\ \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

where

1) Z **is max slack**; 2) $\begin{pmatrix} G_i \\ H_i \end{pmatrix}$ **lin. indep.** 3) $H_m \succeq 0$

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad}) \ \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

where

1) Z **is max slack**; 2) $\begin{pmatrix} G_i \\ H_i \end{pmatrix}$ **lin. indep.** 3) $H_m \succeq 0$

Proof that $(P_{SD,bad})$ is badly behaved:

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad}) \ \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

where

1) Z **is max slack**; 2) $\begin{pmatrix} G_i \\ H_i \end{pmatrix}$ **lin. indep.** 3) $H_m \succeq 0$

Proof that $(P_{SD,bad})$ **is badly behaved:**

x **feas. with slack** S $\Rightarrow$ **last** $n - r$ **cols of** S **are zero**

$\Rightarrow \quad x_{k+1} = \cdots = x_m = 0$

$\Rightarrow \quad \sup -x_m = 0$

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad}) \ \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

where

1) Z **is max slack**; 2) $\begin{pmatrix} G_i \\ H_i \end{pmatrix}$ **lin. indep.** 3) $H_m \succeq 0$

Proof that $(P_{SD,bad})$ is badly behaved:

But: no dual soln with value 0.

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad}) \quad \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

where

1) Z **is max slack**; 2) $\begin{pmatrix} G_i \\ H_i \end{pmatrix}$ **lin. indep.** 3) $H_m \succeq 0$

Proof that $(P_{SD,bad})$ **is badly behaved:**
But: no dual soln with value 0.

Example: before reformulation

$$x_1\begin{pmatrix}54 & 46 & 50 & 4\\ 46 & -38 & 87 & -106\\ 50 & 87 & -60 & 296\\ 4 & -106 & 296 & -368\end{pmatrix}+x_2\begin{pmatrix}110 & 91 & 105 & -6\\ 91 & -72 & 171 & -210\\ 105 & 171 & -72 & 528\\ -6 & -210 & 528 & -672\end{pmatrix}+x_3\begin{pmatrix}42 & 35 & 40 & 0\\ 35 & -28 & 67 & -82\\ 40 & 67 & -36 & 216\\ 0 & -82 & 216 & -27\end{pmatrix}$$

$$+x_4\begin{pmatrix}36 & 30 & 35 & -2\\ 30 & -24 & 57 & -70\\ 35 & 57 & -24 & 176\\ -2 & -70 & 176 & -224\end{pmatrix}\preceq\begin{pmatrix}389 & 323 & 370 & -12\\ 323 & -257 & 610 & -748\\ 370 & 610 & -288 & 1920\\ -12 & -748 & 1920 & -243\end{pmatrix}$$

Hard to tell if well or badly behaved

Example: after reformulation

$$x_1\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)+x_2\left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)+x_3\left(\begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & -1 \\ \hline 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{array}\right)$$

$$+x_4\left(\begin{array}{cc|cc} 0 & 0 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ \hline 3 & 2 & 4 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \preceq \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

As before: $x_3 = x_4 = 0 \Rightarrow \sup -x_4 = 0$

But: no dual solution with value 0

Corollaries:

- **Similar reformulation for well-behaved systems.**

Corollaries:

- Similar reformulation for well-behaved systems.
- The question:

  Is (P_{SD}) well behaved?

  is in $NP \cap coNP$ in real number model of computing.

Corollaries:

- Similar reformulation for well-behaved systems.
- The question:

  Is (P_{SD}) well behaved?

  is in $NP \cap coNP$ in real number model of computing.
- Certificate: reformulation, and proof that Z is max rank slack.

Corollaries:

- Similar reformulation for well-behaved systems.
- The question:

  Is (P_{SD}) well behaved?

  is in $NP \cap coNP$ in real number model of computing.
- Certificate: reformulation, and proof that Z is max rank slack.
- (P_{SD}) well behaved $\Rightarrow$ for all c with a finite obj. value $\exists$ optimal

$$Y = \begin{pmatrix} \overbrace{Y_{11}}^{r} & 0 \\ 0 & Y_{22} \end{pmatrix}$$

How about proving infeasibility?

This part is joint with Minghui Liu.

Semidefinite System (spectrahedron)

$$\begin{aligned} A_i \bullet X &= b_i \, (i = 1, \ldots, m) \qquad\qquad (P) \\ X &\succeq 0 \end{aligned}$$

Semidefinite System (spectrahedron)

$$\begin{aligned} A_i \bullet X &= b_i \, (i = 1, \ldots, m) \qquad\qquad (P) \\ X &\succeq 0 \end{aligned}$$

Here

- A_i are symmetric matrices.
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

Farkas' Lemma for SDP

- $\mathbf{(1) \Rightarrow (2)}$:

(1) $\sum_{i=1}^{m} y_i A_i \succeq 0, \ \sum_{i=1}^{m} y_i b_i = -1 \ (P_{\text{alt}})$ **is feasible.**

(2) $A_i \bullet X = b_i \, \forall i, \ X \succeq 0 \ (P)$ **is infeasible.**

Farkas' Lemma for SDP

- $\mathbf{(1) \Rightarrow (2)}$:

(1) $\sum_{i=1}^m y_i A_i \succeq \mathbf{0}, \ \sum_{i=1}^m y_i b_i = -1$ (P_{alt}) **is feasible.**

(2) $A_i \bullet X = b_i \, \forall i, \ X \succeq 0$ (P) **is infeasible.**

- **Proof: One line.**

Farkas' Lemma for SDP

- **(1) $\Rightarrow$ (2)**:

  (1) $\sum_{i=1}^{m} \boldsymbol{y_i A_i} \succeq \mathbf{0}, \ \sum_{i=1}^{m} \boldsymbol{y_i b_i} = \mathbf{-1}$ $(\boldsymbol{P}_{\text{alt}})$ is feasible.

  (2) $\boldsymbol{A_i} \bullet \boldsymbol{X} = \boldsymbol{b_i} \, \forall \boldsymbol{i}, \ \boldsymbol{X} \succeq \mathbf{0}$ $(\boldsymbol{P})$ is infeasible.

- **Proof: One line.**

- **However: (2) $\not\Rightarrow$ (1)**: $(\boldsymbol{P}_{\text{alt}})$ **is not an exact certificate of infeasibility.**

Literature: exact certificates of infeasibility

- Ramana 1995

- Klep, Schweighofer 2013

- Waki, Muramatsu 2013: variant of facial reduction of

- Borwein, Wolkowicz 1981

- Also: Ramana, Tuncel, Wolkowicz, 1997

Literature: exact certificates of infeasibility

- **Ramana's dual, and certificate of infeasibility: needs $O(n)$ copies of the system, extra variables, and constraints like $U_{i+1} \succeq W_i W_i^T$**

Literature: exact certificates of infeasibility

- Ramana's dual, and certificate of infeasibility: needs $O(n)$ copies of the system, extra variables, and constraints like $U_{i+1} \succeq W_i W_i^T$

- Goal: Find an exact certificate of infeasibility that is "almost" as simple as Farkas' Lemma.

Infeasible example, and proof of infeasibility

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \bullet \boldsymbol{X} = \mathbf{0}$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix} \bullet \boldsymbol{X} = -\mathbf{1}$$

$$\boldsymbol{X} \succeq \mathbf{0}$$

Infeasible example, and proof of infeasibility

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet X = -1$$

$$X \succeq 0$$

- Suppose X feasible $\Rightarrow X_{11} = 0$
  $\Rightarrow X_{12} = X_{13} = 0$
  $\Rightarrow X_{22} = -1$

Infeasible example, and proof of infeasibility

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet X = -1$$

$$X \succeq 0$$

- Suppose X feasible $\Rightarrow X_{11} = 0$
  $\Rightarrow X_{12} = X_{13} = 0$
  $\Rightarrow X_{22} = -1$
- Main idea: We will find such a structure in every infeasible semidefinite system.

Reformulation (again!)

$$\begin{aligned} A_i \bullet X &= b_i \ (i = 1, \ldots, m) \\ X &\succeq 0 \end{aligned} \qquad (\mathbf{P})$$

Reformulation (again!)

$$\begin{aligned} A_i \bullet X &= b_i \ (i = 1, \ldots, m) \\ X &\succeq 0 \end{aligned} \tag{P}$$

- We obtain a reformulation of (P) by a sequence of the following:

(1) Elementery row operations on the equations.

(2) $A_i \leftarrow V^T A_i V \ (i = 1, , \ldots, m)$, where V is invertible.

Reformulation (again!)

$$\begin{aligned} A_i \bullet X &= b_i \ (i = 1, \ldots, m) \\ X &\succeq 0 \end{aligned} \qquad (\mathrm{P})$$

- We obtain a reformulation of (P) by a sequence of the following:

(1) Elementery row operations on the equations.
(2) $A_i \leftarrow V^T A_i V \ (i = 1, , \ldots, m)$, where V is invertible.

- (1) is inherited from Gaussian elimination.

Reformulation (again!)

$$\begin{aligned} A_i \bullet X &= b_i \ (i = 1, \ldots, m) \\ X &\succeq 0 \end{aligned} \qquad \text{(P)}$$

- We obtain a reformulation of (P) by a sequence of the following:

(1) Elementery row operations on the equations.
(2) $A_i \leftarrow V^T A_i V \ (i = 1, , \ldots, m)$, where V is invertible.

- (1) is inherited from Gaussian elimination.

- **Fact:** Reformulations preserve (in)feasibility.

Some linear algebra

$$\begin{pmatrix} \boldsymbol{I_r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bullet \boldsymbol{X} = \mathbf{0},\ \boldsymbol{X} \succeq \mathbf{0} \Rightarrow ?$$

Some linear algebra

$$\begin{pmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \bullet \boldsymbol{X} = 0, \; \boldsymbol{X} \succeq \boldsymbol{0} \Rightarrow \; \boldsymbol{X} = \begin{pmatrix} \boldsymbol{0}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_{22} \end{pmatrix}, \; \boldsymbol{X}_{22} \succeq \boldsymbol{0}.$$

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_{k+1}' \bullet X &= -1 \qquad\qquad\qquad (\mathrm{P_{ref}}) \\ &\vdots \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A_i' look like

$$A_1' = \left(\begin{array}{cc} \overbrace{}^{r_1} & \overbrace{}^{n-r_1} \\ I & 0 \\ 0 & 0 \end{array} \right), \quad A_i' = \left(\begin{array}{ccc} \overbrace{}^{r_1+\ldots+r_{i-1}} & \overbrace{}^{r_i} & \overbrace{}^{n-r_1-\ldots-r_i} \\ \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{array} \right)$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_{k+1}' \bullet X &= -1 \qquad\qquad (\mathrm{P}_{\mathrm{ref}}) \\ &\vdots \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A_i' look like

$$A_1' = \left(\begin{array}{cc} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{array} \right), \ A_i' = \left(\begin{array}{ccc} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{array} \right)$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

Proof of " $\Leftarrow$ " :

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$
\begin{aligned}
A'_i \bullet X &= 0 \; (i = 1, \ldots, k) \\
A'_{k+1} \bullet X &= -1 \qquad\qquad\qquad (\mathrm{P}_{\mathrm{ref}}) \\
&\vdots \\
X &\succeq 0
\end{aligned}
$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A'_i look like

$$
A'_1 = \begin{array}{c} \begin{array}{cc} \overbrace{}^{r_1} & \overbrace{}^{n-r_1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad A'_i = \begin{array}{c} \begin{array}{ccc} \overbrace{}^{r_1+\ldots+r_{i-1}} & \overbrace{}^{r_i} & \overbrace{}^{n-r_1-\ldots-r_i} \end{array} \\ \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \end{array}
$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

Proof of " $\Leftarrow$ " : Suppose that X feasible in $(\mathrm{P}_{\mathrm{ref}})$

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_{k+1}' \bullet X &= -1 \qquad\qquad (\mathrm{P_{ref}}) \\ &\vdots \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A_i' look like

$$A_1' = \begin{array}{cc} \scriptstyle r_1 & \scriptstyle n-r_1 \\ \left(\begin{array}{c} I \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ 0 \end{array}\right) \end{array}, \quad A_i' = \begin{array}{ccc} \scriptstyle r_1+\ldots+r_{i-1} & \scriptstyle r_i & \scriptstyle n-r_1-\ldots-r_i \\ \left(\begin{array}{c} \times \\ \times \\ \times \end{array}\right. & \begin{array}{c} \times \\ I \\ 0 \end{array} & \left.\begin{array}{c} \times \\ 0 \\ 0 \end{array}\right) \end{array}$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

Proof of "$\Leftarrow$" : Suppose that X feasible in $(\mathrm{P_{ref}})$
$\Rightarrow$ first r_1 rows of X are 0

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$
\begin{aligned}
A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\
A_{k+1}' \bullet X &= -1 \qquad\qquad\qquad (\mathrm{P}_{\mathrm{ref}}) \\
&\vdots \\
X &\succeq 0
\end{aligned}
$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A_i' look like

$$
A_1' = \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{r_1} & \overbrace{\quad}^{n-r_1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad A_i' = \begin{array}{c} \begin{array}{ccc} \overbrace{\qquad}^{r_1+\ldots+r_{i-1}} & \overbrace{\quad}^{r_i} & \overbrace{\qquad}^{n-r_1-\ldots-r_i} \end{array} \\ \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \end{array}
$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

Proof of " $\Leftarrow$ " : Suppose that X feasible in $(\mathrm{P}_{\mathrm{ref}})$

$\Rightarrow$ first r_1 rows of X are 0

$\ldots$

$\Rightarrow$ first $r_1 + \ldots + r_k$ rows of X are 0

Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{array}{rcll} A_i' \bullet X & = & 0 \ (i = 1, \ldots, k) & \\ A_{k+1}' \bullet X & = & -1 & \qquad (\mathrm{P}_{\mathrm{ref}}) \\ & \vdots & & \\ X & \succeq & 0 & \end{array}$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A_i' look like

$$A_1' = \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix}, \ A_i' = \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

Proof of "$\Leftarrow$" : Suppose that X feasible in $(\mathrm{P}_{\mathrm{ref}})$

$\Rightarrow$ first r_1 rows of X are 0

$\ldots$

$\Rightarrow$ first $r_1 + \ldots + r_k$ rows of X are 0

$\Rightarrow$ $A_{k+1}' \bullet X \geq 0$

Back to the Example

- Back to the example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet X = -1$$

$$X \succeq 0$$

Back to the Example

- Back to the example:

$$\overbrace{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}}^{A_1'} \bullet X = 0$$

$$\underbrace{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}}_{A_2'} \bullet X = -1$$

$$X \succeq 0$$

Proof outline

- **Based on simplified facial reduction algorithm: construct the A_i' one by one.**

Proof outline

- **Based on simplified facial reduction algorithm: construct the A_i' one by one.**

- **“Difficult” direction is about 1.5 pages.**

Proof outline

- Based on simplified facial reduction algorithm: construct the A_i' one by one.

- "Difficult" direction is about 1.5 pages.

- Alternative: adapt a traditional facial reduction algorithm, the closest one is by Waki and Muramatsu.

Corollary: simple proof that SDP feasibility is in $NP \cap coNP$ in real number model

- **Proof of NP: show feasible X.**

Corollary: simple proof that SDP feasibility is in $NP \cap coNP$ in real number model

- Proof of NP: show feasible X.
- Proof of co-NP: reformulation and how we got it:
  - $V \in \mathbb{R}^{n \times n}$ to encode all similarity transformations.
  - $T \in \mathbb{R}^{m \times m}$ to encode elementary row ops.

Is this just theory?

- **We cannot construct the reformulations in poly time :(**
- **To do so, we would need to solve SDPs exactly.**
- **However...**

Application 1: generating infeasible SDPs

(P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_{k+1}' \bullet X &= -1 \qquad\qquad (P_{\mathrm{ref}}) \\ &\vdots \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A_i' look like

$$A_1' = \begin{array}{c} \begin{array}{cc} \overbrace{}^{r_1} & \overbrace{}^{n-r_1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad A_i' = \begin{array}{c} \begin{array}{ccc} \overbrace{}^{r_1+\ldots+r_{i-1}} & \overbrace{}^{r_i} & \overbrace{}^{n-r_1-\ldots-r_i} \end{array} \\ \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \end{array}$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

Application 1: generating infeasible SDPs
Theorem: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A'_i \bullet X &= 0 \ (i = 1, \dots, k) \\ A'_{k+1} \bullet X &= -1 \qquad\qquad\qquad (\mathrm{P_{ref}}) \\ &\vdots \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, and for $i = 1, \dots, k+1$ the A'_i look like

$$A'_1 = \begin{array}{c} \begin{array}{cc} \overbrace{}^{r_1} & \overbrace{}^{n-r_1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad A'_i = \begin{array}{c} \begin{array}{ccc} \overbrace{}^{r_1+\ldots+r_{i-1}} & \overbrace{}^{r_i} & \overbrace{}^{n-r_1-\ldots-r_i} \end{array} \\ \begin{pmatrix} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix} \end{array}$$

with $r_1, \dots, r_k > 0$, $r_{k+1} \geq 0$.

- Using this result, we can generate all infeasible SDP problems, as:

(1) Generate a system like $(\mathrm{P_{ref}})$.

(2) Reformulate it.

Application 1: generating infeasible SDPs

- **We can generate challenging instances!**
- **Problem library by Liu-P 2016: infeasible and weakly infeasible SDPs**
- **As to solving them: Douglas-Rachford splitting of Liu–Ryu–Yin 2017;**
- **Homotopy method of Hauenstein,Liddell, Zhang 2018**

Application 2: recognizing infeasibility in practice

- **Sometimes we do not even have to reformulate an SDP to find the trivial structure that proves infeasibilty . . . or to reduce the SDP.**

- **Zhu–P–Tran-Dinh Sieve-SDP preprocessor**

Application 2: recognizing infeasibility in practice

- **Sometimes we do not even have to reformulate an SDP to find the trivial structure that proves infeasibilty . . . or to reduce the SDP.**

- **Zhu–P–Tran-Dinh** **Sieve-SDP preprocessor**
- **Before and after picture of an SDP**

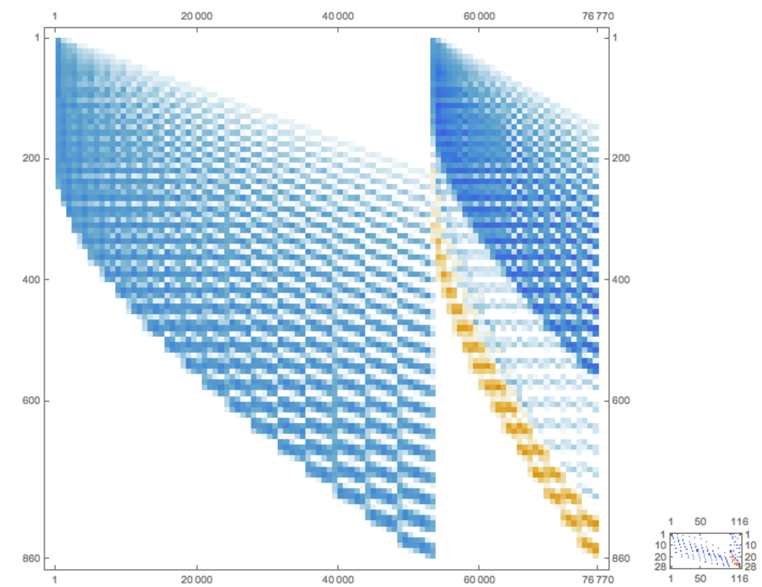

Figure 1: Instance "ex4.2_order20": size and sparsity before and after preprocessing

Some more pathologies: suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$
\begin{array}{ll}
\sup & c_2' x_2 \\
s.t. & x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array} \right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array} \right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \\ \hline \end{array} \right)
\end{array}
$$

where $\Lambda \succ 0, \ M \neq 0, \ c_2' > 0, s \geq 0.$

Papers

- **P:** On the closedness of the linear image of a closed convex cone, Math of OR, 2007
- **P:** Bad semidefinite programs: they all look the same, , SIOPT 2017
- **Liu–P:** Exact duality in semidefinite programming based on elementary reformulations, SIOPT 2015
- **Liu–P:** Exact duals and short certificates fo infeasibility and weak infeasibility in conic linear programming,

  Math. Programming 2017.
- **P:** Characterizing bad semidefinite programs: normal forms and short proofs

  SIAM Review, to appear
- **Zhu–P–Tran-Dinh:** Sieve-SDP: a simple algorithm to preprocess semidefinite programs

  Mathematical Programming Computation, to appear
- **P:** On positive duality gaps in semidefinite programming, 2018

Conclusion

- **Pathologies in duality:** **well- and badly behaved semidefinite (feasible) systems.**

Conclusion

- **Pathologies in duality:** well- and badly behaved semidefinite (feasible) systems.

- Combinatorial type characterizations.

Conclusion

- **Pathologies in duality:** well- and badly behaved semidefinite (feasible) systems.

- Combinatorial type characterizations.

- **Reformulations** to easily recognize good and bad behavior $\rightarrow NP \cap co - NP$ certificates.

Conclusion

- **Pathologies in duality:** **well- and badly behaved semidefinite (feasible) systems.**
- **Combinatorial type characterizations.**
- **Reformulations to easily recognize good and bad behavior $\rightarrow NP \cap co - NP$ certificates.**

Conclusion

- **Pathologies in duality:** well- and badly behaved semidefinite (feasible) systems.

- Combinatorial type characterizations.

- **Reformulations** to easily recognize good and bad behavior $\rightarrow NP \cap co - NP$ certificates.

- Exact, simple certificate of infeasibility of a semidefinite system based on elementary reformulation.

Conclusion

- **Pathologies in duality:** well- and badly behaved semidefinite (feasible) systems.

- Combinatorial type characterizations.

- **Reformulations** to easily recognize good and bad behavior $\rightarrow NP \cap co-NP$ certificates.

- Exact, simple certificate of infeasibility of a semidefinite system based on elementary reformulation.

Conclusion

- **Pathologies in duality:** well- and badly behaved semidefinite (feasible) systems.

- Combinatorial type characterizations.

- **Reformulations** to easily recognize good and bad behavior $\rightarrow NP \cap co - NP$ certificates.

- Exact, simple certificate of infeasibility of a semidefinite system based on elementary reformulation.

- Algorithm to systematically generate **all** infeasible SDPs.

Conclusion

- **Pathologies in duality:** **well- and badly behaved semidefinite (feasible) systems.**
- **Combinatorial type characterizations.**
- **Reformulations to easily recognize good and bad behavior** $\rightarrow NP \cap co - NP$ **certificates.**
- **Exact, simple certificate of infeasibility of a semidefinite system based on elementary reformulation.**
- **Algorithm to systematically generate all infeasible SDPs.**
- **Practical uses: 1) problem library; 2) preprocessing.**

Thank you!