Bad semidefinite programs: they all look the same

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A pair of Semidefinite Programs (SDP)

$$\begin{array}{ll} \sup_x \; c^T x & \qquad \inf_Y \; B \bullet Y \\ \quad s.t. \; \sum_{i=1}^m x_i A_i \preceq B & \qquad\qquad Y \succeq 0 \\ & \qquad\qquad A_i \bullet Y = c_i \, (i = 1, \ldots, m). \end{array}$$

Here

- A_i, B are symmetric matrices, $c, x \in \mathbb{R}^m$.
- $A \preceq B$ means that $B - A$ is symmetric positive semidefinite (psd).
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

Conic LPs and SDPs

• Common framework for LP and SDP: both $\mathbb{R}^n_+$ and psd matrices are closed convex cones.

• A set C is a cone, if $x \in C, \lambda \geq 0 \Rightarrow \lambda x \in C$.

• Linear objective, conic constraint both in LP and SDP, and many other interesting problems, notably SOCPs.

Why is SDP important: applications in

- 0–1 Integer programming.
- Approx algorithms
- Chemical engineering
- Chemistry
- Coding theory
- Control theory
- Combinatorial opt
- Discrete geometry
- Eigenvalue optimization
- Facility planning
- Finance
- Geometric optimization
- Global optimization
- Graph visualization
- Inventory theory
- Machine learning
- Matrix analysis
- PDEs
- Probability theory
- Robust optimization
- Signal processing
- Statistics
- Structural optimization

SDP theory and applications

- **Nice duality theory:** see later
- **Applications:** see textbooks by
  - **Boyd-Vandenberghe**
  - **Ben-Tal-Nemirovskii**
- **Algebraic geometry:**
  - **Nie-Sturmfels 2010**
  - **von Bothmer-Ranestad 2009**
  - **Gouveia, Parrilo, Thomas, 2010**
  - **Book by Blekherman, et al, 2013**
- **Polynomial optimization:**
  - **Lasserre 2000 –**
  - **Parrilo 2000 –**
  - **Nie 2000 –**
  - **Helton-Vinnikov 2003**

SDP duality

The primal-dual pair of SDPs:

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Ideal situation: $\exists \bar{x}, \exists \bar{Y} : c^T \bar{x} = B \bullet \bar{Y}$.

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But: in SDP, unlike in LP pathological phenomena occur: nonattainment, positive gaps.

This is bad, since we would like a certificate of optimality.

Pathology # 1: nonattainment in dual

Primal:

$$\sup \ 2x_1 \quad \text{s.t.} \quad x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \Leftrightarrow \quad \sup \ 2x_1 \quad \text{s.t.} \quad \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0$$

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Here $\inf = 0$, but not attained: Any $y_{11} > 0,\ y_{22} = 1/y_{11}$ is feasible, but $y_{11} = 0$ is not.

Pathology # 2: positive duality gap

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Dual value is 1, and it is attained.

Terminology

Definition:

- **The system $P_{SD} = \{ x \mid \sum_{i=1}^m x_i A_i \preceq B \}$ is well-behaved, if for all c such that**

  $$\sup\{ c^T x \mid x \in P_{SD} \} \text{ is finite,}$$

  the dual program has the same value, and it attains.
- **Badly behaved, otherwise.**
- **We would like to understand well/badly behaved systems.**

Some literature

- **Conic LPs may be badly behaved when K is not polyhedral.**
- **Borwein-Wolkowicz 1981 Facial reduction: theoretical construction of well behaved system**
- **Ramana 1995 Extended dual for SDP**
- **Ramana, Tunçel, Wolkowicz, 1997 Facial reduction implies correctness of extended dual**
- **Klep, Schweighofer, 2013 Related duals based on algebraic geometry.**
- **Waki, Muramatsu, 2013; Pataki 2013: Simpler facial reduction algorithms.**
- **P 2007 Closedness of linear image of a closed, convex cone**

Motivation

The systems

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are both badly behaved.

Curious similarity:

- "Hanging off" diagonals;
- if we delete 2nd row and 2nd column in all matrices in the second system, and delete the first matrix, we get back the first system.

Why all bad SDPs look the same

- **Semidefinite system:**

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- **W.l.o.g. the max (rank) slack is**

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (P_{SD}) badly behaved $\Leftrightarrow$ $\exists V$ a lin. combination of the A_i and B as

$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0,\ \mathrm{R}(V_{12}^T) \not\subseteq \mathrm{R}(V_{22}).$$

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- **Ex:** $x_1 \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{V} \preceq \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^{Z}$

What is missing?

- Matrices Z, V prove that (P_{SD}) is badly behaved.
- But: this is not yet a poly time, or easy to verify proof of bad behavior

Reformulations of

$$(P_{SD}) \ \sum_{i=1}^m x_i A_i \preceq B$$

are obtained by a sequence of:

- Rotate all matrices by $T = \begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix}$, M orthogonal.
- $B \leftarrow B + \sum_{i=1}^m \mu_i A_i$
- $A_i \leftarrow \sum_{j=1}^m \lambda_j A_j$ where $\lambda_i \neq 0$

Reformulations preserve well/badly behaved status; preserve max slack; provide an equivalence relation

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad}) \quad \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

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Proof that $(P_{SD,bad})$ is badly behaved:

x **feas. with slack** $S \ \Rightarrow \ \text{supp}(S) \subseteq \text{supp}(Z)$

$$\Rightarrow \ x_{k+1} = \cdots = x_m = 0 \ \Rightarrow \ \sup -x_m = 0$$

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Proof that $(P_{SD,bad})$ **is badly behaved:**

$$Y \succeq 0, Y \bullet Z = 0 \Rightarrow \qquad Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_{22} \end{pmatrix}$$

$$\Rightarrow \qquad Y \bullet \begin{pmatrix} F_m & G_m \\ G_m^T & H_m \end{pmatrix} \geq 0$$

$\Rightarrow$ **no dual soln with value** 0

Example: before reformulation

$$x_1\begin{pmatrix} 54 & 46 & 50 & 4 \\ 46 & -38 & 87 & -106 \\ 50 & 87 & -60 & 296 \\ 4 & -106 & 296 & -368 \end{pmatrix} + x_2\begin{pmatrix} 110 & 91 & 105 & -6 \\ 91 & -72 & 171 & -210 \\ 105 & 171 & -72 & 528 \\ -6 & -210 & 528 & -672 \end{pmatrix} + x_3\begin{pmatrix} 42 & 35 & 40 & 0 \\ 35 & -28 & 67 & -82 \\ 40 & 67 & -36 & 216 \\ 0 & -82 & 216 & -272 \end{pmatrix}$$

$$+x_4\begin{pmatrix} 36 & 30 & 35 & -2 \\ 30 & -24 & 57 & -70 \\ 35 & 57 & -24 & 176 \\ -2 & -70 & 176 & -224 \end{pmatrix} \preceq \begin{pmatrix} 389 & 323 & 370 & -12 \\ 323 & -257 & 610 & -748 \\ 370 & 610 & -288 & 1920 \\ -12 & -748 & 1920 & -2432 \end{pmatrix}$$

Hard to tell if well or badly behaved

Example: after reformulation

$$x_1 \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + x_2 \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + x_3 \left(\begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & -1 \\ \hline 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{array}\right)$$

$$+ x_4 \left(\begin{array}{cc|cc} 0 & 0 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ \hline 3 & 2 & 4 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \preceq \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

As before: $x_3 = x_4 = 0 \Rightarrow \sup -x_4 = 0$

But: no dual solution with value 0

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where

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- **The question:**

  Is (P_{SD}) well behaved?

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- Corollary: we can generate all linear maps under which the image of the psd cone is closed.

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- **Corollary:** we can generate all well behaved semidefinite systems: choose in sequence H_i, G_i, F_i. Then do reformulation.
- **Corollary:** we can generate all linear maps under which the image of the psd cone is closed.
- **Proof:** $\{(A_i \bullet Y)_{i=1}^m \,|\, Y \succeq 0\}$ is closed $\Leftrightarrow \sum_{i=1}^m x_i A_i \preceq 0$ is well behaved.

Broader framework: Well- and badly behaved conic LPs

- **Conic linear system, with K closed, convex cone:**

$$(P) \quad Ax \leq_K b \qquad\qquad (\Leftrightarrow b - Ax \in K)$$

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where (D_c) is dual program. Badly behaved if not well behaved.

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- **Known:**

  K **polyhedral** $\Rightarrow$ (P) **is well-behaved.**

  (P) **Slater, i.e.,** $\exists x : b - Ax \in \operatorname{ri} K \Rightarrow (P)$ **is well-behaved.**

Given K closed, convex cone

- dual cone: $K^* = \{ y \mid \langle x, y \rangle \geq 0 \, \forall x \in K \}$
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- polyhedral, semidefinite, second order cones are nice.

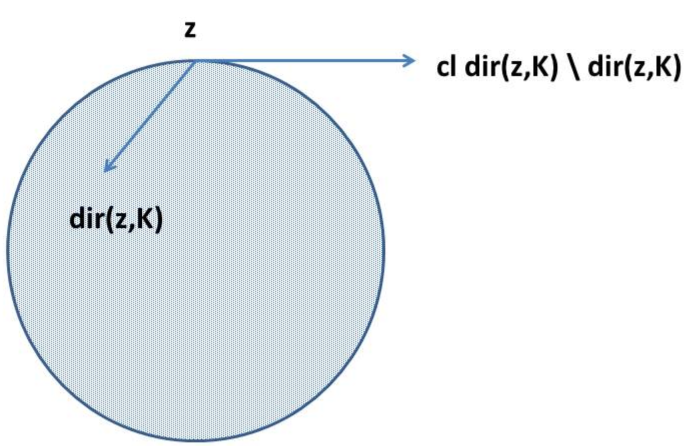
Given K closed, convex cone

- dual cone: $K^* = \{\, y \mid \langle x, y \rangle \geq 0 \, \forall x \in K \,\}$

- K is nice if $K^* + F^\perp$ is closed for all F faces of K.

- polyhedral, semidefinite, second order cones are nice.

- set of feasible directions at $z \in K$ – maybe not closed:
  $\mathrm{dir}(z, K) = \{\, y \mid \exists \epsilon > 0 \text{ s.t. } z + \epsilon y \in K \,\}$



A geometric result

- Conic system:

$$(P) \quad Ax \leq_K b$$

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- K polyhedral, or (P) Slater $(z \in \mathrm{ri} K) \Rightarrow \mathrm{dir}(z,K)$ closed.

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- So, the set

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$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

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$$V \in \mathrm{cl}\ \mathrm{dir}(Z, PSD) \setminus \mathrm{dir}(Z, PSD)$$

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$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0,\ \mathrm{R}(V_{12}^T) \not\subseteq \mathrm{R}(V_{22}).$$

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- We recover characterization of badly behaved semidefinite systems.

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  - Block-diagonality of all dual multipliers
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  - Generating all linear maps under which the image of the psd cone is closed.

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- More generally: conditions for well and badly behaved nature of a conic linear system

- Exact characterization when K is nice.

- Latest version of paper is on Optimization Online.

Thank you!