# Exact duality in semidefinite programming based on elementary reformulations

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# Farkas' Lemma for Linear Programs (LP)

- **Exactly one of the following two systems is feasible:**

(1) $Ax = b, \; x \geq 0$

(2) $y^T A \geq 0, \; y^T b = -1$

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- **Easy direction: One line. Hard direction: One page.**

## Semidefinite System (spectrahedron)

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Here

- $A_i$ are symmetric matrices.
- $A \bullet B = \mathrm{trace}(AB)$.
- $X \succeq 0$ means that $X$ is symmetric positive semidefinite (psd).

# Farkas' Lemma for SDP

- **(1) implies (2):**

**(1)** $\sum_{i=1}^{m} y_i A_i \succeq 0,\ \sum_{i=1}^{m} y_i b_i = -1$ $(P_{\text{alt}})$ **is feasible.**

**(2)** $A_i \bullet X = b_i \, \forall i,\ X \succeq 0$ $(P)$ **is infeasible.**

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# Farkas' Lemma for SDP

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**(2)** $A_i \bullet X = b_i \, \forall i,\ X \succeq 0$ $(P)$ **is infeasible.**

- **Proof: One line.**

- **However: (2) does not imply (1):** $(P_{\text{alt}})$ **is not an exact certificate of infeasibility.**

# Literature: exact certificates of infeasibility

- **Ramana 1995**
- **Ramana, Tuncel, Wolkowicz, 1997**
- **Klep, Schweighofer 2013**
- **Waki, Muramatsu 2013:** variant of facial reduction of
- **Borwein, Wolkowicz 1981**

# Literature: exact certificates of infeasibility

- Ramana's dual, and certificate of infeasibility: needs $O(n)$ copies of the system, extra variables, and constraints like $U_{i+1} \succeq W_i W_i^T$

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- Ramana's dual, and certificate of infeasibility: needs $O(n)$ copies of the system, extra variables, and constraints like $U_{i+1} \succeq W_i W_i^T$

- **Goal:** Find an exact certificate of infeasibility that is "almost" as simple as Farkas' Lemma.

## Infeasible example, and proof of infeasibility

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet X = -1$$

$$X \succeq 0$$

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- Suppose $X$ feasible $\Rightarrow X_{11} = 0$

  $\Rightarrow X_{12} = X_{13} = 0$

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- Suppose $X$ feasible $\Rightarrow X_{11} = 0$
  $\Rightarrow X_{12} = X_{13} = 0$
  $\Rightarrow X_{22} = -1$
- Main idea: We will find such a structure in every infeasible semidefinite system.

# Reformulation

$$\begin{aligned} \boldsymbol{A_i} \bullet \boldsymbol{X} &= \boldsymbol{b_i} \ (\boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{m}) \\ \boldsymbol{X} &\succeq \mathbf{0} \end{aligned} \qquad (\mathbf{P})$$

# Reformulation

$$\begin{aligned} A_i \bullet X &= b_i \ (i = 1, \ldots, m) \\ X &\succeq 0 \end{aligned} \qquad \text{(P)}$$

- We obtain a reformulation of (P) by a sequence of the following:

(1) $(A_j, b_j) \leftarrow (\sum_{i=1}^m y_i A_i, \sum_{i=1}^m y_i b_i)$, where $y \in \mathbb{R}^m, y_j \neq 0$.

(2) Exchange two equations.

(3) $A_i \leftarrow V^T A_i V \ (i = 1, , \ldots, m)$, where $V$ is invertible.

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- **Fact:** Reformulations preserve (in)feasibility.

# Theorem 1: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_{k+1}' \bullet X &= -1 \qquad\qquad\qquad (\mathrm{P_{ref}}) \\ A_i' \bullet X &= b_i' \ (i = k+2, \ldots, m) \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, and for $i = 1, \ldots, k+1$ the $A_i'$ look like

$$A_1' = \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix}, \ A_i' = \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

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$\Rightarrow$ first $r_1$ rows of $X$ are $0$

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**Proof of " $\Leftarrow$ " :** Suppose that $X$ feasible in $(\mathrm{P}_{\mathrm{ref}})$

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...

$\Rightarrow$ first $r_1 + \ldots + r_k$ rows of $X$ are $0$

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$\ldots$

$\Rightarrow$ first $r_1 + \ldots + r_k$ rows of $X$ are $0$

$\Rightarrow A_{k+1}' \bullet X \geq 0$

# Back to the Example

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$$\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{A'_1} \bullet X = \overbrace{0}^{b'_1}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A'_2} \bullet X = \underbrace{-1}_{b'_2}$$

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with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

- It resembles the traditional Farkas' Lemma:
  - The if direction is easy.
  - When $k = 0$, we recover the "usual" Farkas' Lemma.

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with $r_1, \dots, r_k > 0$, $r_{k+1} \geq 0$.

- Using this result, we can generate all infeasible SDP problems, as:

(1) Generate a system like $(\mathrm{P_{ref}})$.

(2) Reformulate it.

# How about feasible systems?

## Theorem 2, Part 1: (P) feasible with maximum rank solution of rank $p \geq 0 \Leftrightarrow$ it has a reformulation:

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_i' \bullet X &= b_i' \ (i = k+1, \ldots, m) \ \ (\mathrm{P_{ref,feas}}) \\ X &\succeq 0 \end{aligned}$$

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with $r_1, \ldots, r_k > 0$, $r_1 + \cdots + r_k = n - p$ and a feasible solution with rank $p$.

## Theorem 2, Part 1: (P) feasible with maximum rank solution of rank $p \geq 0 \Leftrightarrow$ it has a reformulation:

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_i' \bullet X &= b_i' \ (i = k+1, \ldots, m) \ (\mathrm{P}_{\mathrm{ref,feas}}) \\ X &\succeq 0 \end{aligned}$$

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with $r_1, \ldots, r_k > 0$, $r_1 + \cdots + r_k = n - p$ and a feasible solution with rank $p$.

Proof of " $\Leftarrow$ " : Like in the infeasible case.

**Theorem 2, Part 1: (P) feasible with maximum rank solution of rank $p \geq 0 \Leftrightarrow$ it has a reformulation:**

$$\begin{aligned} A'_i \bullet X &= 0 \ (i = 1, \dots, k) \\ A'_i \bullet X &= b'_i \ (i = k+1, \dots, m) \ \ (\mathrm{P_{ref,feas}}) \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, and for $i = 1, \dots, k$ the $A'_i$ look like

$$A'_1 = \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix}, \quad A'_i = \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

with $r_1, \dots, r_k > 0$, $r_1 + \cdots + r_k = n - p$ and a feasible solution with rank $p$.

- Using this result, we can generate all feasible SDPs with max rank soln of rank $p$, as:

(1) Generate a system like $(\mathrm{P_{ref,feas}})$.

(2) Reformulate it.

**Theorem 2, Part 2: Replace $X \succeq 0$ by $X \in 0 \oplus S_+^p$ in $(\mathrm{P_{ref,feas}})$:**

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\begin{aligned}
A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\
A_i' \bullet X &= b_i' \ (i = k+1, \ldots, m) \quad (\mathrm{P_{ref,feas,red}}) \\
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**Then, this system is strictly feasible, i.e., $\exists$ feasible $X \in ri(0 \oplus S_+^p)$**

**Proof: Trivial: some $X$ has rank $p$, and the system proves that no solution can have larger rank.**

**Theorem 2, Part 3: Replace $X \succeq 0$ by $X \in 0 \oplus S^p_+$ in $(\mathrm{P}_{\mathrm{ref,feas}})$:**

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**Then, for all $C \in S^n$ the SDP**

$$\sup\{\, C \bullet X \,|\, X \text{ feasible in } (\mathrm{P}_{\mathrm{ref,feas,red}})\}$$

**has strong duality with its Lagrange dual**

$$\inf\{\, \textstyle\sum_{i=1}^m y_i b_i \,|\, C - \sum_{i=1}^m y_i A_i' \in (0 \oplus S_+^p)^* \}.$$

**(i.e., values agree, and the latter is attained.)**

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(i.e., values agree, and the latter is attained.)

**Proof:** The system $(\mathrm{P_{ref,feas,red}})$ is (trivially) strictly feasible.

# Well behaved systems

- We say that $(\mathrm{P}_{\mathrm{ref,feas,red}})$ is well-behaved, if

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has strong duality with Lagrange dual for all $C$.

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- Characterization of when a system is well behaved (over $\mathbb{S}^n_+$): Pataki, 2011.

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- Latter paper: also an algorithm to generate all well behaved systems.

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- In particular, to generate all linear maps, under which the image of $\mathbb{S}^n_+$ is closed.

# Proof outline

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- **“Difficult” direction is about 1.5 pages.**

- **Alternative: adapt a traditional facial reduction algorithm, the closest one is by Waki and Muramatsu.**

## Context of spectrahedra, and possible other uses

- **In different language: we have a standard form of spectrahedra, to easily check emptiness, or a tight upper bound on the rank of feasible solutions.**
- **Research on spectrahedra:**
  - **Nie-Sturmfels;**
  - **Netzer-Plaumann-Schweighofer;**
  - **Vinzant;**
  - **Blekherman et al;**
  - **Helton-Nie;**
  - **Sinn-Sturmfels; . . .**
- **Will these results be useful in studying spectrahedra?**

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- **For weakly infeasible SDPs, see the talk of Takashi Tsuchiya.**

- **Paper to appear in SIOPT.**

- **For a generalization of our work to general conic LPs; to generate a library of infeasible and weakly infeasible SDPs: followup paper on arxiv, and talk at ISMP.**

# Boldog születésnapot, Tamas!

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**Thank you!**