Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming

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joint work with Minghui Liu

Shinji Mizuno's birthday conference

A primal-dual pair of conic LP

$$
\begin{array}{llllllll}
 & \sup & \langle c, x\rangle & & & \inf & \langle b, y\rangle & \\
(P) & s.t. & Ax & \leq_K b & & s.t. & A^*y = c & (D) \\
 & & & & & & y \geq_{K^*} 0 &
\end{array}
$$

Here

- K **:** **closed convex cone**;
- $K^* = \{y | \langle y, x\rangle \geq 0, \forall x \in K\}$ **dual cone** of K;
- $s \leq_K t \Leftrightarrow t - s \in K$;

A primal-dual pair of conic LP

$$\begin{array}{llll} & \sup \ \langle c, x \rangle & & \inf \ \langle b, y \rangle \\ (P) & s.t. \quad Ax \ \leq_K b & & s.t. \ \ A^*y \ = \ c \ (D) \\ & & & y \ \geq_{K^*} 0 \end{array}$$

- Weak duality is easy: $\langle c, x \rangle \leq \langle b, y \rangle$.
- But: positive gaps, nonattainment can happen.
- “”Usual” Farkas’ lemma for say (D) only works, if

$$\mathrm{dist}(\{y | A^*y = c\}, K^*\}) > 0$$

- (This is called strong infeasibility)
- infeasible, but not strongly = weakly infeasible

Literature:

- Ramana 1995 exact extended SDP dual
- Ramana, Tuncel, Wolkowicz, 1997
- Klep, Schweighofer 2013
- Waki, Muramatsu 2013: variant of facial reduction of
- Borwein, Wolkowicz 1981
- P 2011 Bad semidefinite programs: they all look the same
- P, 2013 simplifed facial reduction, and generalization of Ramana's dual
- Liu, P, 2014 elementary reformulations of SDPs

- **Lourenco et al 2013 analysis of weakly infeasible SDPs**
- **Waki 2012 how to generate weakly infeasible SDPs from poly opt**
- **Permenter, Parrilo, 2014 partial facial reduction**
- **Faulk, P, Tran-Dinh, 2016 simple facial reduction**

Plan: Exact duals and certificates of infeasibility, which

- are "almost" as simple as the usual dual and Farkas' lemma;

- yield basic results in convex analysis;

- yield practical algorithms to generate instances.

- **Remark:** paper deals with duals of (P) and of (D) as well;

- Here we look at (weak) infeasibility of (D)

- **Main idea:** weakly infeasible = infeasible + not strongly infeasible.

Tool 1: Facial Reduction Cone

$$FR_k(K) = \{(y_1, \ldots, y_k) : y_i \in (K \cap y_1^\perp \cap \ldots \cap y_{i-1}^\perp)^*, \forall i\}.$$

Convex cone (!) which is almost as good as K^*:

- $\big((y_1, z_1), \ldots, (y_k, z_k)\big) \in FR_k(K \times C) \Leftrightarrow$
$\big(y_1, \ldots, y_k\big) \in FR_k(K)$ and $\big(z_1, \ldots, z_k\big) \in FR_k(C)$

$$(k = 1\, case : (K \times C)^* = K^* \times C^*)$$

Tool 2: elementary reformulation

$$\begin{array}{llllllll} & \sup \ \langle c, x\rangle & & & \inf \ \langle b, y\rangle & & & \\ (P) & s.t. & Ax & \leq_K b & s.t. & A^*y & = & c \ (D) \\ & & & & & y & \geq_{K^*} & 0 \end{array}$$

• Do elementary row operations on the (D) constraints

$$\langle a_i, y\rangle = c_i \, (i = 1, \ldots, m)$$

• $b \leftarrow b + A\mu$, $\mu \in \mathbb{R}^m$.

• If $K = K^*$ also allow

$$a_i \leftarrow Ta_i \, \forall i, \; b \leftarrow Tb, \text{ where } T \in Aut(K)$$

Theorem: (D) infeasible $\Leftrightarrow$ it has an elem. reformulation:

$$\begin{aligned} \langle a_i', y \rangle &= 0 \, (i = 1, \dots, k) \\ \langle a_{k+1}', y \rangle &= -1 \\ &\vdots \\ y \geq_{K^*} 0 \end{aligned} \qquad (D_{\text{ref}})$$

where $k \geq 0$, $(a_1', \dots, a_{k+1}') \in FR_{k+1}(K^*)$.

Reminder:

$$(a_1', \dots, a_{k+1}') \in FR_{k+1}(K^*) \Leftrightarrow$$

$$a_i' \in (K^* \cap a_1'^{\perp} \cap \dots \cap a_{i-1}'^{\perp})^* \, \forall i$$

Theorem: (D) infeasible $\Leftrightarrow$ it has a reformulation:

$$\begin{array}{rcl} \langle a_i', y \rangle & = & 0 \, (i = 1, \ldots, k) \\ \langle a_{k+1}', y \rangle & = & -1 \\ & \vdots & \\ y \geq_{K^*} 0 & & \end{array} \qquad (D_{\mathrm{ref}})$$

where $k \geq 0$, $(a_1', \ldots, a_{k+1}') \in FR_{k+1}(K^*)$.

$\rightarrow$ **"Row echelon form" of conic LPs**

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Proof of " $\Leftarrow$ **" : Suppose** y **feasible in** $(\mathrm{D}_{\text{ref}})$

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Proof of " $\Leftarrow$ " : Suppose y feasible in $(\mathrm{D}_{\mathrm{ref}})$

$$\Rightarrow y \in K^* \cap a_1'^{\perp} \cap \cdots \cap a_k'^{\perp}$$

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Proof of " $\Leftarrow$ **" : Suppose** y **feasible in** $(\mathrm{D}_{\mathrm{ref}})$

$$\Rightarrow y \in K^* \cap a_1'^{\perp} \cap \cdots \cap a_k'^{\perp}$$

$$\Rightarrow \langle a_{k+1}', y \rangle \geq 0.$$

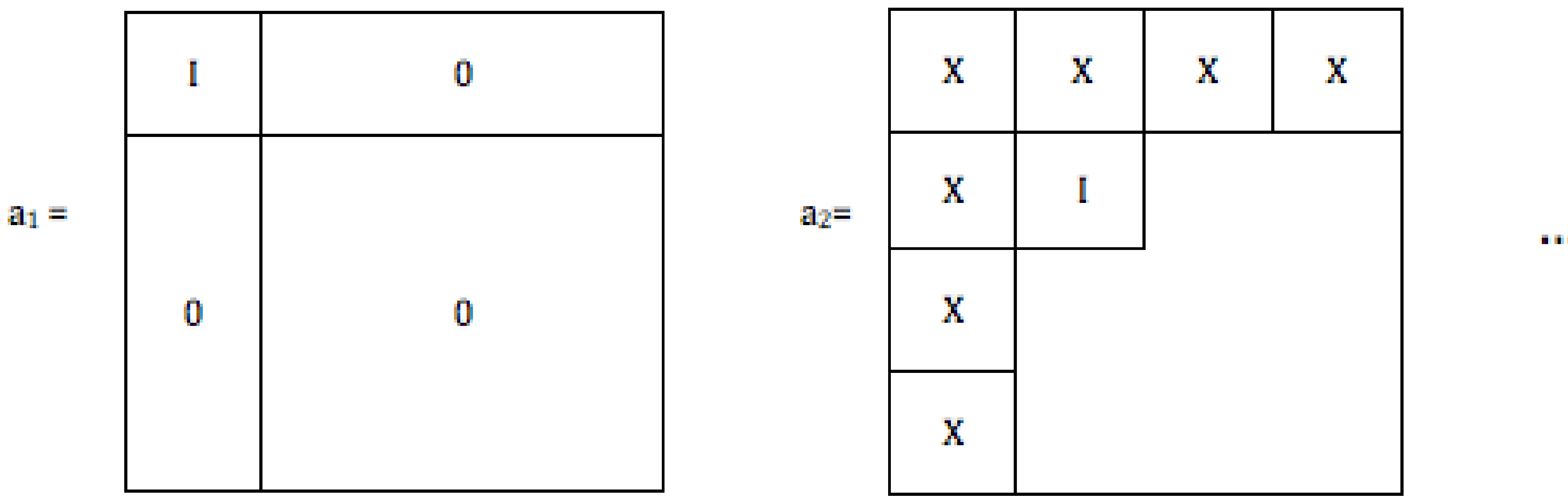
Theorem: (D) infeasible $\Leftrightarrow$ it has a reformulation:

$$\begin{array}{rcl} \langle a_i', y \rangle & = & 0 \, (i = 1, \ldots, k) \\ \langle a_{k+1}', y \rangle & = & -1 \\ & \vdots & \\ y \geq_{K^*} 0 & & \end{array} \qquad (D_{\text{ref}})$$

where $k \geq 0$, $(a_1', \ldots, a_{k+1}') \in FR_{k+1}(K^*)$.

$\rightarrow$ **Can generate all infeasible instances for SDP, and SOCP as:**
generate instances as above, then reformulate

In SDP, we can assume the a_i' to look like



Example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet y = 0$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet y = -1$$

$$y \succeq 0$$

Theorem: (D) is not strongly infeasible $\Leftrightarrow$ there is

$$(y_1, \ldots, y_{\ell+1}) \in FR_{\ell+1}(K) \text{ where } \ell \geq 0$$

such that

$$\begin{aligned} A^* y_j &= 0 \, (j = 1, \ldots, \ell) \\ A^* y_{\ell+1} &= c. \end{aligned}$$

- Special case: if $\ell = 0$, then (D) is feasible.
  $(FR_1(K) = K^*)$

Weakly infeasible = infeasible + not strongly infeasible

$\rightarrow$ **Many nice corollaries.**

Connection to convex analysis

- Classic question: when is the linear image of a closed convex cone closed?
- Rephrased: when is $A^* K^*$ closed?
- Rockafellar; Bauschke-Borwein (1999); Borwein-Moors (2009-10)
- Pataki 2007: some nec., some suff. conditions
- $\rightarrow$ exact characterization, when $K = K^* =$ psd cone
- Origin of "Bad semidefinite programs" paper.

Connection to this work

- $\boldsymbol{A^*K^*}$ is not closed $\Leftrightarrow \exists \boldsymbol{c}$ s.t.

$$\begin{array}{rcl} \boldsymbol{A^* y} & = & \boldsymbol{c} \\ \boldsymbol{y} & \geq_{\boldsymbol{K^*}} & \boldsymbol{0} \end{array} \qquad (\boldsymbol{D})$$

is weakly infeasible.

Connection to this work

- A^*K^* is not closed $\Leftrightarrow \exists c$ s.t.

$$\begin{array}{rcl} A^*y & = & c \\ y & \geq_{K^*} & 0 \end{array} \qquad (D)$$

is weakly infeasible.

More precisely:

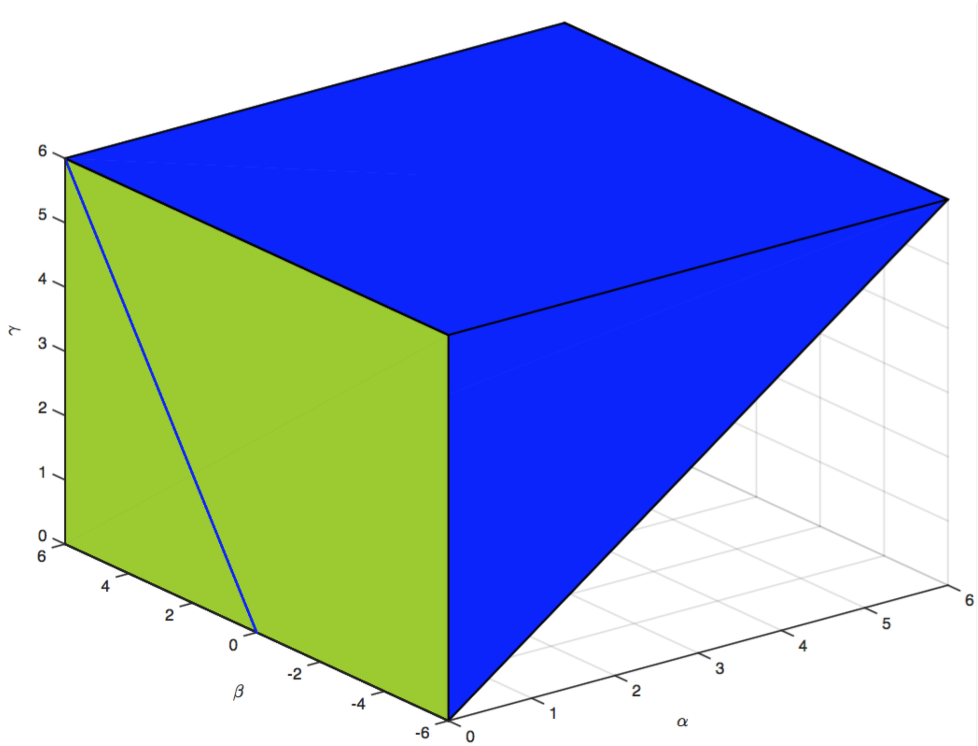
$$\begin{aligned} \text{front}(A^*K^*) &:= \text{cl}(A^*K^*) \setminus A^*K^* \\ &= \{\, c \,|\, (D) \text{ is weakly infeasible } \} \end{aligned}$$

front(A^*K^*) is the frontier of A^*K^*

Example

$$A^* y = (y_{11}, y_{22} + 2y_{13}, y_{11} + y_{22})$$

where y is a symmetric 3×3 matrix



$A^* \mathbb{S}^3_+$ is blue, frontier is green

$\forall c \in \text{front}(A^* \mathbb{S}^3_+)$ we can prove (D) is weakly infeasible.

Theorem A^*K^* **is not closed** $\Leftrightarrow \exists k \geq 1,\ \ell \geq 1$ **s.t.**

- $\exists (a_1, \ldots, a_{k+1}) \in FR_{k+1}(K^*)$ s.t. $a_i \in R(A)\, \forall i$;
- $\exists\, (y_1, \ldots, y_{\ell+1}) \in FR_{\ell+1}(K)$ s.t. $y_j \in N(A^*)\, \forall j \leq \ell$
- s.t.

$$\langle a_i, y_{\ell+1} \rangle = \begin{cases} 0 \text{ if } i \leq k \\ -1 \text{ if } i = k+1. \end{cases}$$

- Easy to see how it subsumes earlier conditions, e.g. P 2007

Theorem A^*K^* **is not closed** $\Leftrightarrow \exists k \geq 1,\ \ell \geq 1$ **s.t.**

- $\exists (a_1, \ldots, a_{k+1}) \in FR_{k+1}(K^*)$ s.t. $a_i \in R(A)\, \forall i;$
- $\exists\, (y_1, \ldots, y_{\ell+1}) \in FR_{\ell+1}(K)$ s.t. $y_j \in N(A^*)\, \forall j \leq \ell$
- s.t.

$$\langle a_i, y_{\ell+1} \rangle = \begin{cases} 0 \text{ if } i \leq k \\ -1 \text{ if } i = k+1. \end{cases}$$

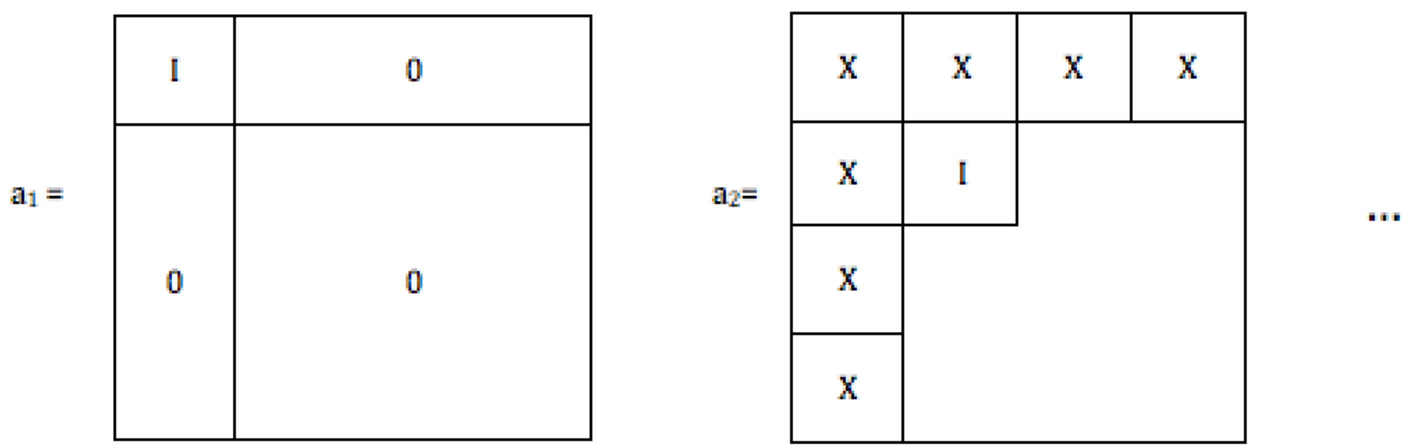
- Characterizes when $K^* + F^\perp$ is not closed, where F is a face of K : take A s.t. $\mathrm{Null}(A^*) = F^\perp$
- Cone K is nice if $K^* + F^\perp$ is closed $\forall F$ faces of K.
- Previous work by Chua-Tuncel; Pataki; Roschina; Roschina-Tuncel

Connection to computational SDP

- **Want to generate challenging infeasible, and weakly infeasible SDPs**

- **Infeasible SDPs: Find A, c s.t. (D) is infeasible in row echelon form.**

- **Also, the a_i' look like:**



Connection to computational SDP

• **Previous work by Waki: generate weakly infeasible instances from Lasserre relaxation for poly opt**

Connection to computational SDP: our work

- **We find weakly infeasible instances, as:**
- **Find $\boldsymbol{A}$, $\boldsymbol{c}$ s.t. $(\boldsymbol{D})$ infeasible in row echelon form.**
- **Find $(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{\ell+1}) \in \boldsymbol{FR}_{\ell+1}(\mathbb{S}^n_+)$ s.t.**

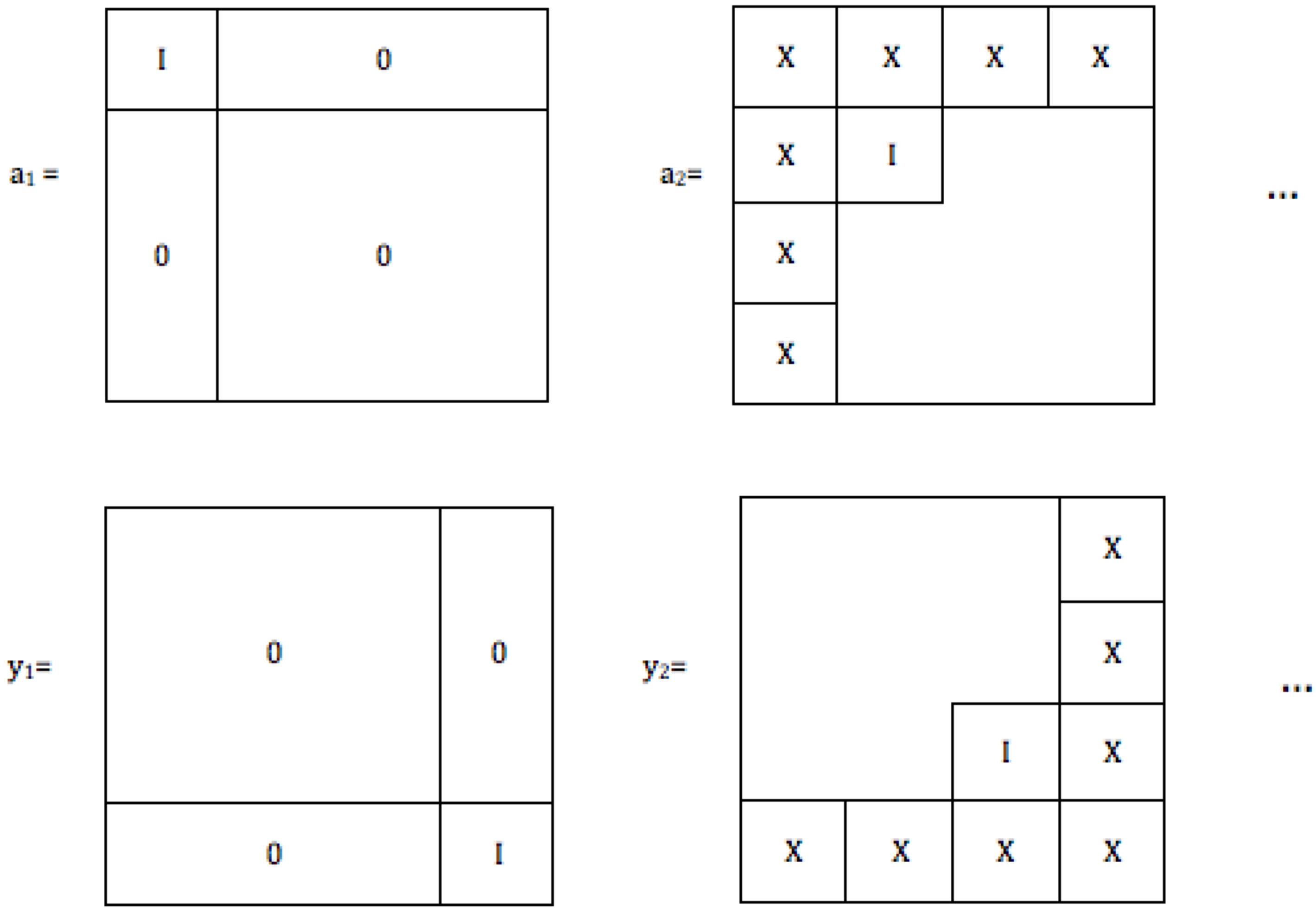
$$\begin{aligned} \boldsymbol{A}^*\boldsymbol{y}_j &= 0 \, (j = 1, \ldots, \ell) \\ \boldsymbol{A}^*\boldsymbol{y}_{\ell+1} &= \boldsymbol{c}. \end{aligned}$$

(to prove it is not strongly infeasible)

- **Difficult in general: bilinear system of equations**

Connection to computational SDP

Easy if:



Computational testing

- **Clean instances: the a_i are as shown before**

- **Messy instances: we applied random row operations and a rotation**

- **PP + SEDUMI =**
  Permenter, Parrilo partial facial reduction + Sedumi

- **"Innocent looking" instances: small numbers, small condition number of constraint matrix.**

- **n=10; m=10; or m=20**

- **we can manually verify infeasibility in exact arithmetic.**

Computational testing

	Infeasible		Weakly Infeasible	
	Clean	Messy	Clean	Messy
SEDUMI	87	27	0	0
SDPT3	10	5	0	0
MOSEK	63	17	0	0
PP+SEDUMI	100	27	100	0

Table 1: Result for instances with $n = 10, m = 10$

	Infeasible		Weakly Infeasible	
	Clean	Messy	Clean	Messy
SEDUMI	100	100	1	0
SDPT3	100	96	0	0
MOSEK	100	100	11	0
PP+SEDUMI	100	100	0	0

Table 2: Result for instances with $n = 10, m = 20$

We report # of successes out of 100

Conclusion: Exact duals and certificates of infeasibility, which

- **are "almost" as simple as the usual dual and Farkas' lemma;**

- **yield basic results in convex analysis;**

- **yield practical algorithms to generate instances.**

- **Key point: weakly infeasible = infeasible + not strongly infeasible**

- **Remark: More material in paper: duals of (P) and of (D) as well;**

Happy birthday! Thank you!