Cone-LP's and Semidefinite Programs: Geometry and a Simplex-type Method

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Abstract. We consider optimization problems expressed as a linear program with a cone constraint. Cone-LP's subsume ordinary linear programs, and semidefinite programs. We study the notions of basic solutions, nondegeneracy, and feasible directions, and propose a generalization of the simplex method for a large class including LP's and SDP's. One key feature of our approach is considering feasible directions as a sum of *two* directions. In LP, these correspond to variables leaving and entering the basis, respectively. The resulting algorithm for SDP inherits several important properties of the LP-simplex method. In particular, the linesearch can be done in the current face of the cone, similarly to LP, where the linesearch must determine only the variable leaving the basis.

1 Introduction

Consider the optimization problem

$$\begin{array}{ll} Min & cx \\ s.t. & x \in K \\ & Ax = b \end{array} \qquad (P)$$

where K is a closed cone in $\mathcal{R}^k$, $A \in \mathcal{R}^{m \times k}$, $b \in \mathcal{R}^m$, $c \in \mathcal{R}^k$. (P) is called a *linear program over a cone*, or a *cone-LP*. It models a large variety of optimization problems; in fact, every convex programming problem can be cast in this form, see ([14], pg. 103). Cone-LP's have been introduced in the fifties as a natural generalization of linear programs; probably the first paper studying duality is due to Duffin [7]. Two interesting special cases are:

- When $K = \mathcal{R}^k_+$ (P) is an ordinary linear program (LP).
- When $k = n(n+1)/2$, $K = \{x \in \mathcal{R}^k$: the symmetric matrix formed of x is positive semidefinite$\}$, we get a *semidefinite program (SDP)*, a problem, that received a lot of attention recently, see e.g. [14], [2], [10].

As shown by Nesterov and Nemirovskii [14], cone-LP's can be solved in polynomial time by interior-point methods, provided K is equipped with an efficiently computable *self-concordant barrier* function. The semidefinite cone does have this property; in recent years, several interior-point methods for SDP have been proposed [2], [11], [22]. It is a natural question, (and one, that could be asked 40 years ago, when Duffin's paper appeared) whether one could generalize the simplex method for solving a reasonably large class of cone-LP's. As the structure of the semidefinite cone has been thoroughly studied, (see e.g. [4]) SDP seems to be a natural candidate. Such a generalization is obviously of interest from a theoretical viewpoint.

Also, semidefinite programming is quickly becoming a tool to solve practical problems, see e.g. [18], [22]. For large-scale problems, the cost of interior-point methods is frequently prohibitive. Since the primal and dual matrix iterates are (by the nature of the algorithm) full rank, one must perform costly, full factorizations at every step. A simplex-type method in which the iterates are low rank, may be able to avoid this drawback. In fact, active set methods for solving eigenvalue-optimization problems (a subclass of SDP) have been used with a reasonable amount of success [6], [15]. However, these methods did not rely on the notions of basic solution, nondegeneracy and extreme rays of the cone of feasible directions, as the simplex method for LP.

Surprisingly, the literature on generalizations of the simplex method for cone-LP's is scant. The only comprehensive work we are aware of is the book of Anderson and Nash [3]; they describe simplex-type methods for several classes of cone-LP's, however, their treatment does not work for finite-dimensional, nonpolyhedral cones, such as the semidefinite cone. First, let us clarify, which are the main features of the simplex method, that one wishes to carry over. Given a basic feasible solution, the simplex method

1. Constructs a complementary dual solution.
2. If this solution is feasible to the dual problem, (i.e. the slack is nonnegative) it declares optimality.
3. If not, it finds a negative component, and constructs an improving extreme ray of the cone of feasible directions.
4. After a linesearch in this direction, it arrives at a new basic solution.

Also, we are allowed to distinguish basic solutions to be "nondegenerate", and "degenerate", and at first assume that our basic solutions encountered during the algorithm are nondegenerate, provided nondegeneracy is a generic property (that is, the set of degenerate solutions is of measure zero in an appropriate model). We can then deal with the degenerate case separately (let's say, using a perturbation argument).

Therefore, when looking for an appropriate generalization, one must answer these questions:

1. How to characterize basic solutions?
2. How to define nondegeneracy, and show that it is indeed a generic property?
3. How to characterize directions emanating at a current solution, which are in a sense extreme?

In this paper we show, how to address these issues. In particular, we define nondegeneracy of a solution by giving a common generalization of a definition of Shapiro [21], and Alizadeh et. al. [1] for SDP, and the usual definition for LP. Of the above three questions, the last seems the least natural. If K is not polyhedral, the cone of feasible directions in (P) at a current solution $\bar{x}$ is usually not even closed, much less does it have extreme rays. There is a simple way to overcome this difficulty. We decompose every feasible direction into the sum of two directions. The first corresponds to moving in the current face of K, until we hit its boundary (corresponding to a variable leaving the basis in LP). The second component takes us to a higher dimensional face of K (corresponding to a variable entering the basis in LP).

The rest of the paper is structured as follows. In Section 2 we introduce the necessary notation and review preliminaries. In Section 3 we derive our results on the geometry of cone-LP's: we study basic solutions, complementarity and nondegeneracy, and feasible directions. In Section 4 we present two algorithms: the first is a purification algorithm (using the terminology of [3]) to construct a basic feasible solution. Finally, we present our simplex-type method. Most proofs in the paper are either simplified, or omitted; full proofs appear in the full-length version.

In our study we consider general cone-LP's. All our results hold when specialized to LP and SDP; for LP, they are well-known. We use the properties of the semidefinite cone, only when necessary. The reason for this is twofold: First, a large part of our results hold for more general cone-LP's. We also want to extract the simple geometric properties of the positive orthant that make the simplex method so successful in solving linear programs.

2 Preliminaries

We shall heavily use tools from convex analysis. The standard reference is the book of Rockafellar [17] (a more introductory level text is [5]). In the following we let K to be a closed cone in $\mathcal{R}^k$. We call the set

$$K^* = \{y : yx \geq 0 \text{ for all } x \in K\}$$

the *polar cone* of K. If K is closed, so is K^* and $K^{**} = K$. We assume that K is *facially exposed*, that is, every face of K is the intersection of K with a supporting hyperplane. Then there is a natural correspondence between the faces of K and K^*. For F a face of K, the *conjugate face of* F is

$$F^\triangle = \{y \in K^* : yx = 0 \text{ for all } x \in F\}$$

Applying conjugacy twice gives back the original face, i.e. $F^{\triangle\triangle} = F$. The faces of K and K^* form a lattice, under the operations $\vee$ and $\wedge$ where $F \vee G$ is the smallest face containing both F and G and $F \wedge G$ is the intersection of F and G. The relative interior of a convex set S is denoted by ri S.

The positive orthant in n-space is denoted by $\mathcal{R}^n$. The set of n by n symmetric matrices is denoted by $\mathcal{S}^n$.

The nullspace and rangespace of a matrix A are denoted by $N(A)$ and $R(A)$. The cone of n by n symmetric, positive semidefinite matrices is denoted by $\mathcal{S}^n_+$.

For simplicity, the elements of $\mathcal{S}_+^n$ are denoted by small letters; however, for $x \in \mathcal{S}_+^n$ we still write $N(x)$ and $R(x)$ for the null- and rangespaces. These two cones are *self-polar*, i.e. $K^* = K$. For the positive orthant, trivially, faces are in one-to-one correspondence with subsets of indices corresponding to 0 components of the vectors in the face, and the conjugate face is associated with the complement subset.

In the semidefinite cone the faces are in one-to-one correspondence with the *subspaces* of $\mathcal{R}^n$. Precisely, F is a face of $\mathcal{S}_+^n$ if and only if

$$F = F(L) \; = \; \{x : x \in \mathcal{S}_+^n \, , R(x) \subseteq L\}$$

for some subspace L of $\mathcal{R}^n$ (see e.g. [4]). For $F(L)$ we have

$$\begin{aligned} \text{ri } F(L) &= \{x : x \in \mathcal{S}_+^n \, , R(x) = L\} \\ \text{lin } F(L) &= \{x : x \in \mathcal{S}^n \, , R(x) \subseteq L\} \end{aligned}$$

Let $r = \dim L$. Then the rank of matrices in $F(L)$ is at most r, and the dimension of $F(L)$ is

$$t(r) := r(r+1)/2$$

($t(r)$ is the r'th "triangular number"). Also, the *conjugate* face corresponds to the subspace *orthogonal* to L, i.e.

$$F^{\triangle} = F(L^{\perp})$$

Note that in $\mathcal{S}_+^n$, the dimensions of faces can take only n distinct values, while the dimension of $\mathcal{S}_+^n$ is $t(n)$. Also, $\dim F + \dim F^{\triangle} = t(r) + t(n-r) \neq t(n) = \dim K$, except in the trivial cases $r = 0$ and $r = n$. These two phenomena (the missing dimensions and the linear hull of the conjugate faces not spanning the whole space) are present in all nonpolyhedral cones.

In the following we shall also consider the the *dual* of (P) defined as

$$\begin{aligned} & Max \; yb \\ & s.t. \quad z \in K^* \\ & \qquad\;\; A^t y + z = c \end{aligned} \qquad (D)$$

It is easy to see, that taking the dual of (D) again we obtain (P). Weak duality between (P) and (D) is easy to prove; to obtain strong duality, when K is not polyhedral, one needs additional assumptions, see e.g. [2], [23], [19] for the case of SDP. We assume in the rest of the paper, that strong duality holds between (P) and (D).

3 Geometry

3.1 Basic solutions

In this subsection we define basic feasible solutions for cone-LP's, and derive several equivalent characterizations. Consider the dual pair of cone-LP's

$$
\begin{array}{llll}
 & Min\ cx & & \\
(P) & s.t. & x \in K & \\
 & & Ax = b &
\end{array}
\qquad\qquad
\begin{array}{llll}
 & Max\ yb & & \\
(D) & s.t. & z \in K^* & \\
 & & A^t y + z = c &
\end{array}
$$

We denote by *Feas(P)* and *Feas(D)* the feasible sets of (P) and (D), respectively.

Definition 3.1 *We call the extreme points of Feas(P) and Feas(D) primal, and dual basic feasible solutions, (bfs's), resp.*

Theorem 3.2 *Let $\bar{x} \in$ Feas(P), F the smallest face of K containing $\bar{x}$. Then the following statements are equivalent.*

1. *$\bar{x}$ is a basic feasible solution.*
2. $\mathcal{N}(A) \cap \mathrm{lin}\ F = \{0\}$.
3. $\bar{x} = F \cap \{x : Ax = b\}$ *and for all F' proper faces of F,* $F' \cap \{x : Ax = b\} = \emptyset$.

Moreover, if $\bar{x}$ is a bfs, then

4. $\dim F \leq m$.

□

When specialized to linear programs, 3. states that $\bar{x}$ is a bfs if and only if the submatrix of A corresponding to nonzero components has of $\bar{x}$ has linearly independent columns. Also, 4. characterizes a bfs as a solution with *minimal support.*

Similarly, one can obtain a characterization of dual basic feasible solutions, as (D) is also a cone-LP (the cone is $\mathcal{R}^m \times K^*$ and the constraint-matrix is (A^t, I)). For brevity, we state only the results corresponding to 2. and 4. above.

Theorem 3.3 *Let $(\bar{y}, \bar{z})$ be a bfs of (D), G the smallest face of K^* that contains $\bar{z}$. Then $(\bar{y}, \bar{z})$ is a basic feasible solution if and only if*

1. $R(A^t) \cap \mathrm{lin}\ G = \{0\}$.

Moreover, if $(\bar{y}, \bar{z})$ is a bfs, then

2. $\dim G \leq k - m$.

□

In the case of LP, in the last statement of Theorems 3.2 and 3.3 we get $\dim F \leq m$ and $\dim G \leq k - m$, resp., i.e. we recover the well-known bound on the number of nonzeros in basic feasible solutions.

In semidefinite programming the bounds on $\dim F$ and $\dim G$ yield a bound on the *rank* of extreme matrices.

Corollary 3.4 *Let $K = \mathcal{S}^n_+$.*

1. *Let $\bar{x}$ be a basic solution of (P),* rank $\bar{x} = r$. *Then r satisfies $t(r) \leq m$.*

2. *Let* $(\bar{y}, \bar{z})$ *be a basic solution of (D),* rank $\bar{z} = s$. *Then* s *satisfies* $t(s) \leq t(n) - m$.

□

Again, notice that in the last statements of Theorems 3.2 and 3.3 we cannot expect equality in general, if K is not polyhedral. The reason is, that not all possible numbers in $\{0, 1, ..., k\}$ appear as the dimension of some face in K.

3.2 Complementarity and nondegeneracy

This section is motivated by the recent papers of Shapiro [21] and Alizadeh et. al. [1], where they define the notion of strict complementarity and nondegeneracy for SDP's and study their properties. We present a simple, common generalization of their definition and the corresponding definitions for LP, and show that most results corresponding to linear programs hold in this more general setting.

Let x and (y, z) be feasible solutions of (P) and (D), respectively. Since $cx = (yA + z)x = yb + zx$, x and (y, z) are both optimal, if and only if $xz = 0$. Therefore, x and z must lie in conjugate faces of K and K^* respectively. We introduce the following

Definition 3.5 The pair x and (y, z) is *strictly complementary*, if

$$\text{(SC) } x \in \text{ri } F \quad \text{and } z \in \text{ri } F^{\triangle}$$ □

In LP, strict complementarity requires that the sum of the number of nonzeros in the primal and dual slacks be equal to n. In SDP, it requires that the sum of the *ranks* of the primal and dual slack matrices be n. Contrary to the case of LP, in general cone-LP's a strictly complementary solution-pair may not always exist. A counterexample for *SDP* is given in ([1]).

Definition 3.6 *Let* x *be feasible for* (P), F *the smallest face of* K *that contains* x. *We say that*

$$\textit{(PND) } x \textit{ is } \text{nondegenerate } \textit{if } \mathrm{R}(A^t) \cap \text{lin } F^{\triangle} = \{0\}$$

Notice, that the above definition of nondegeneracy can be formally obtained from the characterization of a bfs in 3. of Theorem 3.2 by replacing $N(A)$ by $R(A^t)$ and F by $F^{\triangle}$; extremity and nondegeneracy are complementary notions. In linear programs, the above definition requires the submatrix of A corresponding to nonzero components of x to have linearly independent *rows*. Nondegeneracy of a dual feasible solution can be defined in a similar manner. Exactly as in LP, we get

Theorem 3.7 *Let* x *and* (y, z) *be optimal solutions of* (P) *and* (D), *resp.*

1. *If* x *is nondegenerate, then* (y, z) *is basic.*
2. *Suppose that* x *and* (y, z) *satisfy (SC). Then* x *is basic if and only if* (y, z) *is nondegenerate.*

□

As it is well-known, nondegeneracy is a *generic* property in LP's, i.e. a random vertex of a randomly generated polyhedron is nondegenerate with probability one. As recently shown in [1], a similar property is true for SDP's: a randomly chosen extreme point of a random SDP is nondegenerate with probability one.

If x is a nondegenerate basic solution, then

$$\begin{aligned} &\dim F \leq m \\ &\dim F^{\triangle} \leq k - m \end{aligned} \tag{3.3}$$

In the case of LP, these two bounds imply $\dim F = m$, i.e., as known, the number of nonzeros in a nondegenerate basic solution must be *exactly* m. If $K = \mathcal{S}^n_+$, (3.3) gives upper and lower bounds on the rank of x (observed also in [1]). If rank $x = r$, then (3.3) is equivalent to

$$\begin{aligned} &t(r) \leq m \\ &t(n-r) \leq t(n) - m \end{aligned} \tag{3.4}$$

These bounds allow a range of possible values of r. For example, $n = 10$, $m = 15$ implies $2 \leq r \leq 5$.

In linear programs with a special structure, an upper bound on the number of the nonzeros frequently yields combinatorial results. Similarly, using the upper bound on the extreme ranks in structured SDP's one obtains interesting corollaries, that we may group together under the name *semidefinite combinatorics.* Several examples (detailed in the full-length paper) are

1. A lower bound on the multiplicity of critical eigenvalues in eigenvalue-optimization, see [16].
2. A lower bound on the number of tight constraints in quadratic programs, where both the constraints and the objective function are convex.
3. The polynomial-time solvability of (possibly nonconvex) quadratic programs with few constraints.
4. An upper bound on the dimension of optimal orthonormal representations in the θ_1 and θ_4 formulations in the Lovász theta-function.

3.3 Feasible directions

Consider the primal problem

$$\begin{aligned} &Min\ cx \\ &s.t. \quad x \in K \\ &\qquad\quad Ax = b \end{aligned} \tag{P}$$

For F, a face of K define the set

$$D_F = \{(f, g) : f \in \text{lin } F, g \in K, A(f + g) = 0\}$$

The following simple lemma is crucial.

Lemma 3.8 *Let x be a feasible solution to (P), and F the smallest face of K that contains x. Then d is a feasible direction for x if and only if $d = f + g$ for some $(f, g) \in D_F$.*

Proof (If) Suppose $d = f + g$ for some $(f, g) \in D_F$. Since $f \in \text{lin } F$, $x + \alpha f \in F$ for some $\alpha > 0$, and clearly $(x + \alpha f) + \alpha g \in K$.

(Only if) Suppose $x + \alpha d = g \in K$ for some $\alpha > 0$. Then $d = \frac{1}{\alpha} g + \frac{1}{\alpha}(-x)$, and clearly $\frac{1}{\alpha} g \in K$, $\frac{1}{\alpha}(-x) \in \text{lin } F$. □

Remark 3.9 When $K = \mathcal{R}^n_+$, in the decomposition of d f corresponds to moving in the current face of the positive orthant, while g corresponds to a direction that moves into a higher dimensional face. The cone of feasible directions in the usual sense is a projection of D_F. When K is polyhedral, so is D_F, and also its projection. This is no longer true, when K is not polyhedral. In fact, as shown recently by Ramana et. al. [19], when $K = \mathcal{S}^n_+$ and F is a face, the set

$$\{f + g : f \in \text{lin } F, g \in K\}$$

is *never* closed ! (Of course, its intersection with the constraints $A(f+g) = 0$ *may* be closed; however, this result shows, that an approach considering directions in the projected cone is hopeless in general.) However, as we shall see, when designing a simplex method, it suffices to consider the (f, g) pairs in D_F, there is no need to project.

Theorem 3.10 *Let x and F be as in Lemma 3.8. Then the following hold.*

1. *If x is a bfs, then D_F has extreme rays.*
2. *Let $(f, g) \in D_F$, and G the smallest face of K that contains g. Let*

   $$\alpha^* = \max\{\alpha : x + \alpha(f + g) \in K\}$$

   Then $F \vee G$ contains the feasible segment $[x, x + \alpha^*(f + g)]$ and it is a minimal face of K with this property.
3. *Assume that K satisfies the following property.*

   Property 1 *Let F and G be faces of K. Assume $y \in \text{lin } F \setminus F, g \in G$, $y + g \in K$. Then $F \wedge G \neq \emptyset$.*

   Let (f, g) be an extreme ray of D_F, and define α^* as above. Then

   $$\alpha^* = \max\{\alpha : x + \alpha f \in F\}$$

Proof of 1 D_F is clearly closed. To show that it has extreme rays, it suffices to prove that $D_F \cap (-D_F) = \{0\}$. For simplicity, assume $K \cap (-K) = \{0\}$, and let $(f, g) \in D_F \cap (-D_F)$. Then $g \in K \cap (-K)$ hence $g = 0$. Therefore $f \in \text{lin } F \cap \mathcal{N}(A)$, and since x is basic, $f = 0$.

□

Remark 3.11 Property 1 looks somewhat artificial, hence it is worth checking its validity, when $K = \mathcal{S}^n_+$. Suppose that $F = F(L)$ and $G = F(J)$, where L and J are subspaces of $\mathcal{R}^n$. The condition $y \in \text{lin } F \setminus F$ is satisfied if and only if $R(y) \subseteq L$, but y is not psd. As $y + g$ *is* psd, the rangespace of g must have a nontrivial intersection with $R(y)$.

Remark 3.12 Suppose (f,g) is an extreme ray of D_F. It is natural to ask, whether g must be an extreme ray of K. The answer is *no* in general. Suppose that G is the smallest face of K that contains g, and let us calculate an upper bound on $\dim G$. Similarly to Theorem 3.2 we obtain

$$\dim G \leq m - \dim F + 1 \tag{3.6}$$

(as D_F is defined by $(k - \dim F + m$ constraints, the number of unconstrained variables is k, and (f,g) must be in a face of dimension 1 of D_F). First, let us consider the case of LP. Here $\dim F$ is equal to the number of nonzero components of x. If $\dim F = m$, i.e. x is nondegenerate, then we can move away from x by increasing the value of only one nonbasic variable (this may eventually decrease the value of a basic variable to zero). If x is degenerate, i.e. $\dim F < m$, then we may have to increase the values of $m - \dim F + 1$ variables from zero to a positive value in order to move away from x. In SDP, as explained earlier, we cannot expect $m = \dim F$, even in the nondegenerate case. For instance, consider an SDP with $m = n$. Such semidefinite programs do arise in practice, e.g. the max-cut relaxation SDP, whose feasible set has been termed the *elliptope* and studied in depth by Laurent and Poljak [13]. Then, if a basic solution x is nondegenerate, and of rank r, then r must satisfy

$$t(r) \leq n,\, t(n-r) \leq t(n) - n \Rightarrow 1 \leq t(r) \leq n$$

Thus $t(r) - n$ can be of order n, hence the best upper bound we can give on the rank of g in an extreme ray (f,g) is of order $\sqrt{2n}$. Therefore, in SDP's we cannot expect to be able to move away from the current solution by increasing the rank by 1, i.e. "bringing a one-dimensional subspace into the basis".

4 Algorithms

The following assumption will remain in force throughout the rest of the paper.

Assumption 4.1 *1. Given $x \in K$ one can find generators for* lin F *and* lin $F^{\triangle}$, *where F is the smallest face of K that contains x.*

2. The separation problem *is solvable for the polar cone K^*. Namely, given $z \in \mathcal{R}^k$ we can either*

(a) Assert $z \in K^*$ OR

(b) Assert $z \notin K^*$, and find $v \in K$, satisfying $vz < 0$.

Assumption 4.1 obviously holds for the positive orthant. For $\mathcal{S}_+^n$, 1. can be done in polynomial time by determining the rangespace of a positive semidefinite matrix. Separation for $\mathcal{S}_+^n$, (recall, that $\mathcal{S}_+^n$ is self-polar) can also be done in polynomial time as described e.g. in [9].

Remark 4.2 For simplicity, we shall assume, that all computations are done in exact arithmetic.

4.1 Finding a basic feasible solution

Theorem 4.3 *The following algorithm finds a basic feasible solution of (P) in finitely many iterations.*

Algorithm 1.

Input: $A,\ b,\ c,\ x \in K$ satisfying $Ax = b$.

Output: $\bar{x} \in K$ satisfying $A\bar{x} = b$ and $c\bar{x} \leq cx$.

1. Let F be the smallest face of K that contains x.
2. Find $f \neq 0$ s.t

$$\begin{aligned} Af &= 0 \\ f &\in \text{lin } F \end{aligned} \tag{4.7}$$

   If no such f exists, set $\bar{x} = x$ and STOP.

3. If $cf > 0$, set $f = -f$. Determine

$$\begin{aligned} \alpha^* = &\ max\ \alpha \\ &s.t \quad x + \alpha f \in F \end{aligned} \tag{4.8}$$

4. Set $x = x + \alpha^* f$, and go to 1.

Proof The correctness of Algorithm 1. follows from Theorem 3.2. Since the dimension of the smallest face containing the current x strictly decreases in every iteration, finiteness is obvious. □

4.2 A Simplex-type Method

In this section we first state our simplex procedure. We assume, that we are given x, a basic, nondegenerate solution of (P).

First, we need the following lemma.

Lemma 4.4 *Let x be a feasible solution to (P), and F the smallest face of K that contains x. Then d is in the* closure *of feasible directions for x if and only if $d = f + g$ for some $f \in (\text{lin } F^{\triangle})^{\perp}, g \in K$. Moreover, if g is in the relative interior of K, then d is a feasible direction.*

□

Remark 4.5 Since in a general cone-LP lin F and lin $F^{\triangle}$ together do not span the whole space, the cone of feasible directions is not closed in general. Adding a $g \in \text{ri } K$ to f amounts to perturbing f towards the interior of K.

Algorithm Simplex

Input: $A,\ b,\ c,\ K,\ x \in K$ s.t. $Ax = b$.

1. **Construct a complementary dual solution.** Let F be the smallest face of K containing x, $F^{\triangle}$ is the conjugate of F. Construct subspaces $M_P,\ M_D$ satisfying
   i. $N(A) \oplus (\text{lin } F + M_P) = \{0\}$
   ii. $\mathcal{R}(A^t) \oplus (\text{lin } F^{\triangle} + M_D) = \{0\}$

Find (y, z) s.t.

$$\begin{aligned} z &\in \text{lin } F^{\triangle} + M_D \\ A^t y + z &= c \end{aligned} \tag{4.9}$$

2. **Check optimality.** If $z \in K^*$ STOP; x and (y, z) are optimal. Else, find $g \in K$ s.t. $zg < 0$.
3. **Find improving direction.** Construct (f, h), an extreme ray of D_F s.t. $zh < 0$.
4. **Line-search.** Determine

$$\begin{aligned} \alpha^* = & \; max \; \alpha \\ & s.t \quad x + \alpha f \in F \end{aligned} \tag{4.10}$$

Replace x by $x + \alpha^*(f + h)$ and goto 1.

The difference, as opposed to the LP-simplex method is, that we cannot obtain an extreme ray of D_F in one step. We can proceed as follows.

First, perturb g found in Step 2. to obtain $g_1 \in \text{ri } K$, $zg_1 < 0$. Solve the system

$$\begin{aligned} f &\in \text{lin } F \oplus M_P, \\ A(f + g_1) &= 0 \end{aligned} \tag{4.11}$$

Notice, that in this system the number of variables is the same as the number of constraints. This would *not* be true, if we replaced lin $F \oplus M_P$ by lin F. By Lemma 4.4 $f + g_1$ is a feasible direction, therefore we can find $x_1 = x + \epsilon(f + g_1) \in K$ for some small $\epsilon > 0$. Then $(x_1 - x, x_1) \in D_F$, although it is in general not an extreme ray. However, we can use an algorithm similar to Algorithm 1. to convert it into one.

The main difference between our approach and the LP-simplex method is, that we may not get an extreme ray of D_F in one step. It can be shown, that in the case of SDP, one does not have to choose g_1 to be in ri K, that is of full rank. The rank of g_1 needed to get $(f, g_1) \in D_F$ depends on the dimension of the subspace M_P. The smaller the dimension of M_P, the smaller the rank of g_1 needs to be. If $\dim M_P = 0$, g_1 can be chosen to be equal to g, and no purification is needed to get an extreme ray of D_F.

We conjecture that when the current solution x is basic, and (f, g) is an extreme ray of D_F, then the new solution will also be basic, at least in the case of SDP. So far, we haven't been able to prove this conjecture. What we can state is an upper bound on the rank of the new solution. Clearly the rank of the new solution is at most rank x + rank $h - 1$, and both the rank of x and h can be bounded from above, as x is basic, and (f, h) is an extreme ray of D_F.

5 Conclusion

In this work we presented a generalization of the simplex method for a class of cone-LP's, including semidefinite programs. The main structural results we needed to derive, were

– A characterization of basic solutions.
– Defining nondegeneracy, and deriving some properties of nondegenerate solutions, (building on the results of [21], [1]).
– Characterizing extreme feasible directions in an appropriate higher dimensional space.

These structural results are of some independent interest.
It seems that the success of a simplex-type method for solving SDP's will depend on two issues.

– Its convergence properties.
– An efficient implementation. The advantage of our method, as opposed to an interior-point algorithm is, that our matrices, since they are basic solutions, are low rank. Also, when we move along an extreme ray of D_F the rangespace of the current iterate does not change by much. Therefore, it may be possible to design an efficient update scheme analogous to the update scheme of the revised simplex method for LP.

These issues will be dealt with in the full version of this article.

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