ON THE GENERIC PROPERTIES OF CONVEX OPTIMIZATION PROBLEMS IN CONIC FORM

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Abstract

We prove that strict complementarity, primal and dual nondegeneracy of optimal solutions of convex optimization problems in conic form are generic properties. In this paper, we say generic to mean that the set of data possessing the desired property (or properties) has the same Hausdorff measure as the set of data that does not. Our proof is elementary and it employs an important result due to Larman [7] on the boundary structure of convex bodies.

Keywords: Convex optimization, strict complementarity, nondegeneracy

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1 Introduction

Nondegeneracy and strict complementarity of optimal solutions of optimization problems are very important concepts. Many results in optimization, especially those regarding local convergence properties of interior-point algorithms, have the underlying assumption that there exists a strictly complementary pair of (primal-dual) optimal solutions (see, for instance, some of the recent papers concerning the local convergence properties of interior-point methods on semidefinite programming and related problems: Kojima, Shida and Shindoh [6], Luo, Sturm and Zhang [10], and Potra and Sheng [13]). Even some global complexity results and condition measures sometimes are formulated in terms of certain properties, and magnitudes of strictly complementary solutions.

The purpose of this paper is to give a simple proof that strict complementarity, primal and dual nondegeneracy of optimal solutions of convex optimization problems in conic form are *generic* properties. In this paper, we say *generic* to mean that the set of data possessing the desired property (or properties) has the same Hausdorff measure as the set of data that does not. Our proof is rather elementary, however, it relies on an important result, due to Larman [7]. Our results generalize the corresponding theorems of Alizadeh-Haeberly-Overton [1] and Shapiro [18] for semidefinite programming. The proofs by Alizadeh-Haeberly-Overton and Shapiro use the notion of transversality from algebraic topology. Renegar [16] suggested a generalization of these results to convex optimization problems which are semi-algebraic.[1] Our results also apply to convex optimization problems that cannot be expressed as semi-algebraic formulae.

[1]Renegar's argument uses semi-algebraic geometry: One devises a semi-algebraic formula which is satisfied precisely by data possessing the mentioned property. Thus the set of all data possessing the property is a semi-algebraic set. One then proves that the interior of the set is empty. Semi-algebraic sets with empty interior are always of measure zero.

2 Notation, Definitions and Basic Results

We consider the primal-dual pair of convex optimization problems in the conic form:

$$(P) \qquad \inf \ \langle c, x \rangle$$

$$x \quad \in \quad (L + b) \cap K.$$

We define the dual of (P) as follows:

$$(D) \qquad \inf \ \langle s, b \rangle$$

$$s \quad \in \quad (L^{\perp} + c) \cap K^*,$$

where $L \subset \mathbb{R}^n$ is a subspace, $L^{\perp}$ is its orthogonal complement, $b, c \in \mathbb{R}^n$ are given, and $K \subset \mathbb{R}^n$ is a pointed, closed, convex, solid (with non-empty interior) cone (for this formulation of a primal-dual pair and its properties, see for instance, [11]). K^* is the dual of K under the given inner-product. That is,

$$K^* \ := \ \{s \in \mathbb{R}^n : \ \langle s, x \rangle \ \geq \ 0, \ \forall\, x \in K\}.$$

So, for a fixed convex cone K and an inner product $\langle \cdot, \cdot \rangle$, the data points are defined to be the triples (L, b, c). By a *proper face* of K, we mean a face of K that is neither empty nor equal to K. We will further assume that cone K is facially exposed. (That is, every proper face of K can be expressed as the intersection of K with one of its supporting hyperplanes.) Thus, from now on, when we refer to a face of K we mean an exposed face of K. Since K is a pointed, closed, convex, facially exposed, solid cone, so is its dual, K^*. We say that the problem instance defined by the data (L, b, c) is *gap free* if there exist feasible solutions $\bar{x}$ and $\bar{s}$ for (P) and (D) respectively, such that $\langle \bar{s}, \bar{x} \rangle = 0$. Note that our definition of *gap free* is equivalent to what is usually referred to as "both primal and dual problems attain their optimum values and the duality gap is zero."

Later, we will remove the assumption that K (and hence K^*) is facially exposed, see Remark 4.1.

In the rest of this section, we give a brief summary of the definitions of basic, nondegenerate and strictly complementary solutions, and of their properties. To make our paper self-contained, we also provide several proofs. The results quoted here appear in [12].

Definition 2.1 *The extreme points of the primal and dual feasible solution sets are called primal, and dual basic feasible solutions, respectively.*

Throughout this paper, for a pointed closed convex cone K, $\text{int}(K)$ denotes the interior of K, ∂K denotes the boundary of K except the unique extreme point $\{0\}$. For each non-empty face F of K, lin (F) denotes the linear hull of F, ri (F) denotes the relative interior of F, $\text{rel}\partial(F)$ denotes the relative boundary of F, and $F^{\triangle}$ denotes the *conjugate* of face F and is defined as

$$F^{\triangle} \;:=\; \{s \in K^* :\; \langle s, x\rangle \;=\; 0,\; \forall\, x \in F\}.$$

The conjugate face of a face G of K^* is defined analogously, and also denoted by $G^{\triangle}$. If F is a face of K, then $F^{\triangle\triangle}$ is the smallest exposed face of K that contains F ([2], page 43). Therefore, a face F is exposed iff $F^{\triangle\triangle} = F$.

Theorem 2.1 *Let x be a feasible solution of (P) and F be the smallest face of K containing x. Then x is a basic feasible solution if and only if*

$$L \cap \text{lin}\ (F) = \{0\}.$$

Proof. x is basic iff

$$L \cap \{y :\; x \pm ty \in K \text{ for some} \, t > 0\} \;\;=\;\; \{0\}. \tag{1}$$

F, the smallest face of K that contains x is characterized by $x \in \text{ri}\ (F)$. By elementary convex analysis, the second set on the left hand side of (1) is exactly lin (F), thus our claim follows. □

Definition 2.2 *A pair of feasible primal-dual solutions (x, s) is* strictly complementary, *if*

(SC) $x \in \text{ri}\ F$ and $s \in \text{ri}\ F^{\triangle}$

for some face F of K. If $(0, s)$ is feasible for $s \in int(K^*)$ or $(x, 0)$ is feasible for $x \in int(K)$ then the corresponding pair is also called strictly complementary.

Definition 2.3 *Let x be a feasible solution of (P) and F be the smallest face of K that contains x. We say that*

x *is* nondegenerate *if* $L^\perp \cap \text{lin } (F^\triangle) = \{0\}$.

Notice, that the above definition of nondegeneracy can be formally obtained from the characterization of a basic feasible solution in Theorem 2.1 by replacing L by $L^\perp$ and F by $F^\triangle$; extremity and nondegeneracy are complementary notions. As in the fundamentals of linear programming problems, we obtain

Theorem 2.2 *Let x be a feasible solution of (P). Then the following hold.*

1. *If x is nondegenerate, then an arbitrary complementary solution of (D) must be basic. Therefore, if there is a complementary dual solution, it must be unique.*
2. *Suppose that s is a dual solution and (x, s) satisfies (SC). Then s is basic if and only if x is nondegenerate.*

Proof. Let s be a complementary solution of (D), and G the smallest face of K^* that contains s. By Theorem 2.1 s is basic, iff

$$L^\perp \cap \text{lin } (G) \quad = \quad \{0\}. \tag{2}$$

Since $\langle s, x \rangle = 0$, and $x \in \text{ri } (F)$, we get $s \in F^\triangle$. Then ri $G \cap F^\triangle \neq \emptyset$, so by Theorem 18.1 in [14] $G \subseteq F^\triangle$. Therefore 1. is proved.

If (SC) holds, then $G \cap \text{ri } F^\triangle \neq \emptyset$ (as it contains s), so again by Theorem 18.1 in [14] $G = F^\triangle$. Hence 2. follows as well. □

These definitions and theorems are applicable to all convex optimization problems in conic form. When we take K to be the cone of $n \times n$ symmetric positive semidefinite matrices over the reals, the above definitions and theorems specialize to Semidefinite Programming problems. In this special case, the above definitions and theorems coincide with the corresponding definitions and theorems of Alizadeh-Haeberly-Overton [1].

3 Hausdorff measure, Hausdorff dimension and Larman's Theorem

The foundations of Hausdorff measures can be traced back to the proposal of outer measures in Carathéodory's work around 1915. A useful way of thinking about Hausdorff measures is to think of them as a coverability measure of sets.

For any non-empty subset U of $\mathbb{R}^d$, the *diameter* of U is

$$\text{diam }(U) := \sup\{\|x-y\| : \ x, y \in U\}.$$

Let $S \subseteq \mathbb{R}^d$. A countable (or finite) collection of sets of diameter at most δ, say $\{U_i\}$ is called a *δ-cover of S*, if

$$S \subseteq \bigcup_{i=1}^{\infty} U_i \quad \text{and} \quad 0 < \text{diam }(U_i) \leq \delta, \ \forall i.$$

Let $t \geq 0$ be a real number. Then the t-dimensional Hausdorff measure of S is defined as

$$H^t(S) := \nu(t) \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\text{diam }(U_i)/2)^t : \ \{U_i\} \text{ is a } \delta\text{-cover of } S \right\},$$

where

$$\nu(t) := \frac{\pi^{t/2}}{\Gamma\left(\frac{t}{2}+1\right)}.$$

The concept of Hausdorff measure is quite rich. In general, many different functions, other than the t-power function in the above definition, give rise to interesting Hausdorff measures (see [15]).

When $t \in \mathbb{Z}_{++}$, $\nu(t)$ is the Euclidean volume of the Euclidean unit ball in $\mathbb{R}^t$. If S is a d-dimensional polytope then $H^d(S)$ is also equal to the Euclidean volume of S.

For a fixed set S, there exists unique $t(S) \geq 0$ such that

$$H^t(S) = +\infty \text{ for all } t < t(S) \quad \text{and} \quad H^t(S) = 0 \text{ for all } t > t(S).$$

$t(S)$ is called *the Hausdorff dimension of* S. Moreover, if S is Lebesgue measurable with respect to d-dimensional Lebesgue measure ($d \in \mathbb{Z}_{++}$) then $H^d(S)$ coincides with the Lebesgue d-measure of S.

The notions of Hausdorff measure and Hausdorff dimension are very useful in quantifying sets of Lebesgue measure zero which are nevertheless "substantial." That is, since the Hausdorff dimension can be a real number, it can distinguish, for instance, among various sets all of which may have Lebesgue d-measure 0, and Lebesgue $(d-1)$-measure $+\infty$. So, in this sense, Hausdorff measure is more precise than the Lebesgue measure. These properties are used extensively together with Fubini's Theorem in the next section. In this paper, $\dim(S)$ denotes the Hausdorff dimension of S as defined above.

Our proof is based on certain properties of the boundaries of convex sets. An excellent introduction to the boundary structure of convex sets is given in the early chapters of the book by Schneider [17]. For an excellent introduction to Hausdorff measures, see Rogers' book [15]. We employ an important theorem due to Larman [7]:

Theorem 3.1 *Let* W *be the union of the relative boundaries of those proper faces of a* d*-dimensional compact convex set* S *which have dimension at least one. Then* W *has Hausdorff dimension less than* $(d-1)$.

For example, if S is a 3-dimensional cube, then W is its skeleton, which has zero 2-dimensional Hausdorff measure. (At this point, it should be clear to the reader that the consequences of the above theorem for polytopes are rather trivial.) If S is a sphere, then W is empty.

Larman and many others used the notion of a *cap* of a compact convex set to measure certain subsets of the boundary of the convex compact set. A *cap* of the compact convex solid set S in $\mathbb{R}^d$ is a d-dimensional subset of S that can be written as the intersection of S with a closed halfspace in $\mathbb{R}^d$. In Larman's proof, the extreme points of S are covered by certain caps and then it is shown that the union of these caps has finite measure with respect to the $(d-1)$-dimensional Hausdorff measure. Larman then analyzes the higher dimensional faces of S (which must contain line segments) using some new technical lemmas and some other technical results of [3, 8, 9].

4 Main Results

We denote the set of all proper faces of K, excluding the trivial face $\{0\}$, by $\mathcal{F}(K)$. For a convex compact set S, $\mathcal{F}(S)$ denotes all proper faces of S. We define

$$V := \left\{(x,s) \in \partial K \oplus \partial K^* : \ x \in \text{ri } (F), s \in \text{ri } (F^\triangle) \text{ for some } F \in \mathcal{F}(K)\right\}$$

and

$$\bar{V} := \left\{(x,s) \in \partial K \oplus \partial K^* : \ x \in F, s \in F^\triangle \text{ for some } F \in \mathcal{F}(K)\right\}.$$

We obtain the following key result as a consequence of Theorem 3.1.

Corollary 4.1 *The set $\bar{V} \backslash V$ has zero $\dim(\bar{V})$-dimensional Hausdorff measure.*

Proof. Let $\hat{s} \in \text{int}(K^*)$. The set

$$K' = \{x \in K : \ \langle \hat{s}, x\rangle = 1\}$$

is a convex compact set of dimension $(n-1)$ (see, e.g., [4]). The boundary structure of K' completely represents that of K (except for the extreme point 0 of K). The correspondence is: F' is a proper face of K' iff

$$F' = F \cap \{x : \ \langle \hat{s}, x\rangle = 1\}$$

for some face $F \in \mathcal{F}(K)$. We apply Theorem 3.1 to K' to conclude that the set

$$\{x \in K' : \ x \in \text{rel}\partial(F') \text{ for some } F' \in \mathcal{F}(K'), \ \dim(F') > 0\}$$

has Hausdorff dimension less than $(n-2)$. Therefore, the sets

$$\{x \in K : \ x \in \text{rel}\partial(F) \text{ for some } F \in \mathcal{F}(K), \ \dim(F) > 1\} \text{ and}$$

$$\{x \in K : \ x \in \text{rel}\partial(F) \text{ for some } F \in \mathcal{F}(K)\} \tag{3}$$

have Hausdorff dimension less than $(n-1)$. Note that the above two sets differ only in the union of relative boundaries of those faces of K which have dimension 1. These faces are the extreme rays, and the union of their relative boundaries is $\{0\}$.

Similarly, the set

$$\{s \in K^* : \; s \in \mathrm{rel}\partial(F) \text{ for some } F \in \mathcal{F}(K^*)\} \tag{4}$$

has Hausdorff dimension less than $(n-1)$.

Next, we observe that

$$\begin{aligned} \bar{V}\backslash V \;\; \subset \;\; & \left\{(x,s) \in \bar{V} : \; x \in \mathrm{rel}\partial(F), s \in F^{\triangle} \text{ for some } F \in \mathcal{F}(K)\right\} \\ & \cup \left\{(x,s) \in \bar{V} : \; x \in F^{\triangle}, s \in \mathrm{rel}\partial(F) \text{ for some } F \in \mathcal{F}(K^*)\right\}. \end{aligned} \tag{5}$$

Now, note that the projection of the first set in the right hand side of (5), onto the space of K is exactly the set described by (3); similarly, the projection of the second set in the right hand side of (5), onto the space of K^* is precisely the set described by (4). We also note that the corresponding projections of $\bar{V}$ onto K and K^* are ∂K and $\partial(K^*)$ respectively. Clearly, $\dim(\partial K)$ and $\dim(\partial(K^*))$ are both at least $(n-1)$. Now, we apply Fubini's Theorem (the version of it described as Theorem A in page 147 of Halmos' book [5]). We conclude,

$$\dim\left(\left\{(x,s) \in \bar{V} : \; x \in \mathrm{rel}\partial(F), s \in F^{\triangle} \text{ for some } F \in \mathcal{F}(K)\right\}\right) \; < \; \dim(\bar{V})$$

and

$$\dim\left(\left\{(x,s) \in \bar{V} : \; x \in F^{\triangle}, s \in \mathrm{rel}\partial(F) \text{ for some } F \in \mathcal{F}(K^*)\right\}\right) \; < \; \dim(\bar{V}).$$

Since the union of the last two sets covers $\bar{V}\backslash V$, we arrive at $\dim(\bar{V}\backslash V) < \dim(\bar{V})$. □

Next, we relate the sets V and $\bar{V}$ to that set of data points for which the corresponding optimization problems (P) and (D) have interesting properties. To this end, we define

$$C \; := \; V \cup (\{0\} \oplus \mathrm{int}(K^*)) \cup (\mathrm{int}(K) \oplus \{0\}),$$

$$\bar{C} \; := \; \bar{V} \cup (\{0\} \oplus K^*) \cup (K \oplus \{0\}).$$

Now, we are ready to define the data points.

$$\bar{\mathcal{D}}(L) \; := \; \left\{(L, \bar{x}+u, \bar{s}+v) : \; (\bar{x},\bar{s}) \in \bar{C}, \, u \in L, \, v \in L^{\perp}\right\}.$$

Proposition 4.1 *For each subspace* $\{0\} \subset L \subset \mathbb{R}^n$, $\bar{\mathcal{D}}(L)$ *is the set of all instances* (L,b,c) *which are gap free.*

Proof. Let $(L, b, c) \in \bar{\mathcal{D}}(L)$. Then by the definition of $\bar{\mathcal{D}}(L)$, there exist a face F of K and $\bar{x} \in F$, $\bar{s} \in F^{\triangle}$ such that $\bar{x}$ and $\bar{s}$ are feasible solutions of (P) and (D) respectively. By the definition of the conjugate face, $\langle \bar{s}, \bar{x} \rangle = 0$. Thus, the instance (L, b, c) is gap free. Conversely, suppose that the given triple (L, b, c) is gap free. Then there exist $\hat{x} \in (L + b) \cap K$ and $\hat{s} \in (L^{\perp} + c) \cap K^*$ such that $\langle \hat{s}, \hat{x} \rangle = 0$. If $\hat{x} \notin \partial K$ then all feasible solutions of (P) are optimal. So, without loss of generality, we can assume $\hat{x} \in \partial K$. Let $F \in \mathcal{F}(K)$ such that $\hat{x} \in \text{ri } (F)$ (the remaining case $F = \{0\}$ is trivial, since in this case, all feasible solutions of (D) are optimal). Now, all optimal solutions s of (D) must satisfy $\langle s, \hat{x} \rangle = 0$. Thus, $\hat{s} \in F^{\triangle}$. Since $\hat{x}$ and $\hat{s}$ are feasible in respective problems, we also have $b = \hat{x} + u$ and $c = \hat{s} + v$ for some $u \in L$, $v \in L^{\perp}$. Therefore, $(L, b, c) \in \bar{\mathcal{D}}(L)$. □

Next, we define a subset of $\bar{\mathcal{D}}(L)$.

$$\mathcal{D}(L) \ := \ \left\{ (L, \bar{x} + u, \bar{s} + v) : \ (\bar{x}, \bar{s}) \in C, \, u \in L, \, v \in L^{\perp} \right\}.$$

Proposition 4.2 *For each subspace $\{0\} \subset L \subset \mathbb{R}^n$, $\mathcal{D}(L)$ is the set of all gap free instances (L, b, c) such that the corresponding pair (P) and (D) has strictly complementary pair of solutions.*

Proof. Similar to the proof of Proposition 4.1. □

Using Corollary 4.1, we arrive at the following fact.

Proposition 4.3 *For each subspace $\{0\} \subset L \subset \mathbb{R}^n$, the set $\bar{\mathcal{D}}(L) \backslash \mathcal{D}(L)$ has zero $\dim(\bar{\mathcal{D}}(L))$-dimensional Hausdorff measure.*

Now, we are ready to define

$$\bar{\mathcal{D}} \ := \bigcup_{\{0\} \subset L \subset \mathbb{R}^n} \bar{\mathcal{D}}(L). \tag{6}$$

By Proposition 4.1, $\bar{\mathcal{D}}$ is the set of all data points (excluding the trivial choices for L) such that the underlying problems (P) and (D) have zero duality gap. Similarly, we define

$$\mathcal{D} := \bigcup_{\{0\}\subset L\subset\mathbb{R}^n} \mathcal{D}(L). \tag{7}$$

By Proposition 4.2, $\mathcal{D}$ is the set of all data points (excluding the trivial choices for L) such that the underlying problem pairs (P) and (D) are gap free and have strictly complementary solutions.

To get the final result, we invoke Fubini's Theorem. In our setting, for every fixed choice of L the set of instances violating strict complementarity has measure zero; thus, Fubini's theorem implies the following theorem.

Theorem 4.1 *The set $\bar{\mathcal{D}}\backslash\mathcal{D}$ has zero $\dim(\bar{\mathcal{D}})$-dimensional Hausdorff measure.*

Since we have K and K^* as solid cones in $\mathbb{R}^n$, in the cases $L = \{0\}$ and $L = \mathbb{R}^n$, strict complementarity always holds. So combining Propositions 4.1, 4.2 and Theorem 4.1, we conclude that strict complementarity is one of the generic properties of these convex optimization problems in conic form.

A remarkable result due to Ewald, Larman and Rogers (see [3], or Theorem 2.3.1 in [17]) implies that for fixed L and b, the set of all c, $\|c\|_2 = 1$, for which (P) has multiple optimal solutions has zero $(d-1)$-dimensional Hausdorff measure (here d is the dimension of the feasible region of (P)). Note that all allowable c's, in this context, have positive $(d-1)$-dimensional Hausdorff measure (the surface area of the $d-$hypersphere). Thus, for every fixed pair (L, b), the uniqueness of the optimal solution (under the restriction of existence) holds everywhere in the sense of Hausdorff measure. Again applying Fubini's Theorem, we conclude that the primal uniqueness is generic. Similarly, the dual uniqueness is generic.

Now, we apply part 2 of Theorem 2.2. Let (x, s) be a feasible pair satisfying (SC). Since x is unique almost everywhere, it is basic almost everywhere (whenever there is an optimal solution, there exists one that is also an extreme point of the feasible region). We conclude that s is nondegenerate almost everywhere. The same arguments, starting with the statement that s is unique almost everywhere, establish that the primal nondegeneracy is generic. We proved the following:

Theorem 4.2 *Strict complementarity, primal and dual nondegeneracy of optimal solutions are generic properties of convex optimization problems in conic form.*

Remark 4.1 *Note that every proper face of K that is not exposed, lies on the boundary of some proper exposed face of K. Thus, the arguments above go through for convex cones with faces that are not exposed.*

Remark 4.2 *Note that in computing $\bar{\mathcal{D}}$ and $\mathcal{D}$ in equations (6), (7) we can restrict the union to those subspaces L with dimension m. That is, we define*

$$\bar{\mathcal{D}}_m := \bigcup_{L:\dim(L)=m} \bar{\mathcal{D}}(L), \qquad \mathcal{D}_m := \bigcup_{L:\dim(L)=m} \mathcal{D}(L).$$

The same arguments as above imply that $\bar{\mathcal{D}}_m \backslash \mathcal{D}_m$ has zero $\bar{\mathcal{D}}_m$-dimensional Hausdorff measure and that strict complementarity, primal and dual nondegeneracy are generic properties in this context as well.

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