Combinatorial characterizations in semidefinite programming duality

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Partly joint work with Minghui Liu

Talk at Danish Technical University

A pair of Semidefinite Programs (SDP)

$$\sup_x \; c^T x \qquad\qquad \inf_Y \; B \bullet Y$$

$$s.t. \; \sum_{i=1}^m x_i A_i \preceq B \qquad\qquad Y \succeq 0$$

$$A_i \bullet Y = c_i \; (i = 1, \ldots, m).$$

Here

- A_i, B are symmetric matrices, $c, x \in \mathbb{R}^m$.
- $A \preceq B$ means: $B - A$ is positive semidefinite (psd).
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

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- $A \preceq B$ means: $B - A$ is positive semidefinite (psd).
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.
- $Y \succeq 0 \overset{\text{def}}{\Leftrightarrow}$ all principal subdeterminants are nonnegative.
- Equivalently, if $v^T Y v \geq 0 \, \forall v \in \mathbb{R}^n$.

Why is SDP important:
LP $\subseteq$ SDP $\subseteq$ Convex Optimization

LP as SDP:

- If A_i and B are diagonal $\Rightarrow$ so is $B - \sum_{i=1}^{m} x_i A_i$.
- So it is psd iff diagonal elements are nonnegative.
- So LP can be modeled as SDP: make A_i, B diagonal.

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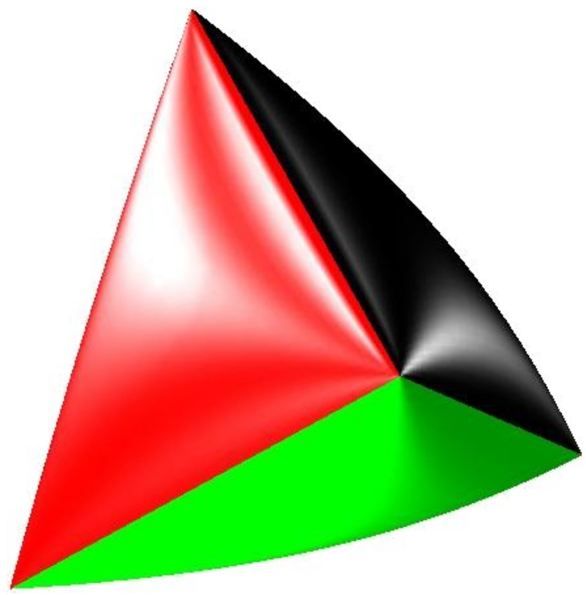
SDP is a convex problem:

- Feasible set is convex, since set of psd matrices is.

3 by 3 correlation matrices

The set $\{ (x, y, z) \mid \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \}$



Why is SDP important: applications in

- 0–1 Integer programming.
- Approx algorithms
- Chemical engineering
- Chemistry
- Coding theory
- Control theory
- Combinatorial opt
- Discrete geometry
- Eigenvalue optimization
- Facility planning
- Finance
- Geometric optimization
- Global optimization
- Graph visualization
- Inventory theory
- Machine learning
- Matrix analysis
- PDEs
- Probability theory
- Robust optimization
- Signal processing
- Statistics
- Structural optimization
- Several thousand papers on SDP in the last 10 years.

SDP duality

The primal-dual pair of SDPs:

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Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

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But: in SDP, unlike in LP pathological phenomena occur: nonattainment, positive gaps.

This is bad, since we would like a certificate of optimality.

Pathology # 1: nonattainment in dual

Primal:

$$\sup\ 2x_1 \quad \Leftrightarrow \quad \sup\ 2x_1$$

$$s.t.\ x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad s.t. \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0$$

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Unattained $\inf = 0 : \; y_{11} > 0$ **is feasible,** but $y_{11} = 0$ **is not.**

Pathology # 2: positive duality gap

Primal:

$$\sup \ x_2$$

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Dual value is 1, and it is attained.

Terminology

Definition:

- **The system**

$$(P_{SD}) \ \sum_{i=1}^{m} x_i A_i \preceq B$$

is well-behaved, if for all c such that

$$\sup\{ \, c^T x \mid x \in P_{SD} \, \} \text{ is finite,}$$

the dual program has the same value, and it attains.

- **Badly behaved, otherwise.**
- **We would like to understand well/badly behaved systems.**

Motivation

The systems

$$x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

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Curious similarity – of these, and about 20 others in the literature

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- **Semidefinite system:**

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$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (P_{SD}) badly behaved $\Leftrightarrow$ $\exists V$ a lin. combination of the A_i and B as

$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0,\ \mathrm{R}(V_{12}^T) \not\subseteq \mathrm{R}(V_{22}).$$

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- **Ex:** $x_1 \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{V} \preceq \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^{Z}$

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- Aside: how do we prove that $Ax = b$ is infeasible? $\rightarrow$ row echelon form.
- We will borrow ideas from the row echelon form to produce easy-to-verify certificates.

Reformulations of

$$(P_{SD}) \ \sum_{i=1}^m x_i A_i \preceq B$$

are obtained by a sequence of:

- Apply a rotation $V^T()V$ to all matrices, where V is invertible.
- $B \leftarrow B + \sum_{i=1}^m \mu_i A_i$
- $A_i \leftarrow \sum_{j=1}^m \lambda_j A_j$ where $\lambda_i \neq 0$

Reformulations preserve well/badly behaved status; preserve max slack; provide an equivalence relation

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad}) \quad \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

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Proof that $(P_{SD,bad})$ is badly behaved:

$$\begin{aligned} x \text{ feas. with slack } S &\Rightarrow \text{ last } n-r \text{ cols of } S \text{ are zero} \\ &\Rightarrow x_{k+1} = \cdots = x_m = 0 \\ &\Rightarrow \sup -x_m = 0 \end{aligned}$$

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Example: before reformulation

$$x_1\begin{pmatrix}54 & 46 & 50 & 4\\ 46 & -38 & 87 & -106\\ 50 & 87 & -60 & 296\\ 4 & -106 & 296 & -368\end{pmatrix}+x_2\begin{pmatrix}110 & 91 & 105 & -6\\ 91 & -72 & 171 & -210\\ 105 & 171 & -72 & 528\\ -6 & -210 & 528 & -672\end{pmatrix}+x_3\begin{pmatrix}42 & 35 & 40 & 0\\ 35 & -28 & 67 & -82\\ 40 & 67 & -36 & 216\\ 0 & -82 & 216 & -272\end{pmatrix}$$

$$+x_4\begin{pmatrix}36 & 30 & 35 & -2\\ 30 & -24 & 57 & -70\\ 35 & 57 & -24 & 176\\ -2 & -70 & 176 & -224\end{pmatrix}\preceq\begin{pmatrix}389 & 323 & 370 & -12\\ 323 & -257 & 610 & -748\\ 370 & 610 & -288 & 1920\\ -12 & -748 & 1920 & -2432\end{pmatrix}$$

Hard to tell if well or badly behaved

Example: after reformulation

$$x_1 \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + x_2 \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + x_3 \left(\begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & -1 \\ \hline 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{array}\right)$$

$$+ x_4 \left(\begin{array}{cc|cc} 0 & 0 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ \hline 3 & 2 & 4 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \preceq \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

As before: $x_3 = x_4 = 0 \Rightarrow \sup -x_4 = 0$

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- **The question:**

  Is (P_{SD}) well behaved?

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  is in $NP \cap coNP$ in real number model of computing.
- Certificate: reformulation, and proof that Z is max rank slack.
- (P_{SD}) well behaved $\Rightarrow$ for all c with a finite obj. value $\exists$ optimal

$$Y = \begin{pmatrix} \overbrace{Y_{11}}^{r} & 0 \\ 0 & Y_{22} \end{pmatrix}$$

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- **Corollary: we can generate all well behaved semidefinite systems: choose in sequence H_i, G_i, F_i. Then do reformulation.**

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- Corollary: we can generate all linear maps under which the image of the psd cone is closed.

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- Corollary: we can generate all well behaved semidefinite systems: choose in sequence H_i, G_i, F_i. Then do reformulation.
- Corollary: we can generate all linear maps under which the image of the psd cone is closed.
- Proof: $\{ (A_i \bullet Y)_{i=1}^m \,|\, Y \succeq 0 \}$ is closed $\Leftrightarrow$ $\sum_{i=1}^m x_i A_i \preceq 0$ is well behaved.

How about proving infeasibility?

This part is joint with Minghui Liu.

Semidefinite System (spectrahedron)

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Here

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- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

Farkas' Lemma for SDP

- **(1) $\Rightarrow$ (2)**:

(1) $\sum_{i=1}^{m} y_i A_i \succeq 0, \; \sum_{i=1}^{m} y_i b_i = -1 \; (P_{\text{alt}})$ **is feasible.**

(2) $A_i \bullet X = b_i \, \forall i, \; X \succeq 0 \; (P)$ **is infeasible.**

Farkas' Lemma for SDP

- **(1) $\Rightarrow$ (2)**:

(1) $\sum_{i=1}^{m} y_i A_i \succeq \mathbf{0}$, $\sum_{i=1}^{m} y_i b_i = -1$ (P_{alt}) **is feasible.**

(2) $A_i \bullet X = b_i \; \forall i$, $X \succeq 0$ (P) **is infeasible.**

- **Proof: One line.**

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(2) $A_i \bullet X = b_i \, \forall i, \ X \succeq 0 \ (P)$ is infeasible.

- **Proof: One line.**

- **However: (2) $\not\Rightarrow$ (1): (P_{alt}) is not an exact certificate of infeasibility.**

Literature: exact certificates of infeasibility

- Ramana 1995
- Klep, Schweighofer 2013
- Waki, Muramatsu 2013: variant of facial reduction of
- Borwein, Wolkowicz 1981
- Also: Ramana, Tuncel, Wolkowicz, 1997

Literature: exact certificates of infeasibility

- **Ramana's dual, and certificate of infeasibility: needs $O(n)$ copies of the system, extra variables, and constraints like $U_{i+1} \succeq W_i W_i^T$**

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- Ramana's dual, and certificate of infeasibility: needs $O(n)$ copies of the system, extra variables, and constraints like $U_{i+1} \succeq W_i W_i^T$

- Goal: Find an exact certificate of infeasibility that is "almost" as simple as Farkas' Lemma.

Infeasible example, and proof of infeasibility

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet X = -1$$

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- Suppose X feasible $\Rightarrow X_{11} = 0$
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  $\Rightarrow X_{22} = -1$
- Main idea: We will find such a structure in every infeasible semidefinite system.

Reformulation (again!)

$$\begin{aligned} A_i \bullet X &= b_i \ (i = 1, \ldots, m) \\ X &\succeq 0 \end{aligned} \qquad (\mathbf{P})$$

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- **We obtain a reformulation of (P) by a sequence of the following:**

(1) Elementery row operations on the equations.
(2) $A_i \leftarrow V^T A_i V \ (i = 1, , \ldots, m)$, where V is invertible.

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- **Fact:** Reformulations preserve (in)feasibility.

Theorem 1: (P) infeasible $\Leftrightarrow$ it has a reformulation

$$\begin{aligned} A_i' \bullet X &= 0 \ (i = 1, \ldots, k) \\ A_{k+1}' \bullet X &= -1 \qquad\qquad (\mathrm{P_{ref}}) \\ &\vdots \\ X &\succeq 0 \end{aligned}$$

where $k \geq 0$, **and for** $i = 1, \ldots, k+1$ **the** A_i' **look like**

$$A_1' = \begin{pmatrix} \overbrace{I}^{r_1} & \overbrace{0}^{n-r_1} \\ 0 & 0 \end{pmatrix}, \ A_i' = \begin{pmatrix} \overbrace{\times}^{r_1+\ldots+r_{i-1}} & \overbrace{\times}^{r_i} & \overbrace{\times}^{n-r_1-\ldots-r_i} \\ \times & I & 0 \\ \times & 0 & 0 \end{pmatrix}$$

with $r_1, \ldots, r_k > 0$, $r_{k+1} \geq 0$.

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$\Rightarrow$ first r_1 rows of X are 0

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where $k \geq 0$, and for $i = 1, \ldots, k+1$ the A'_i look like

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$\Rightarrow A_{k+1}' \bullet X \geq 0$

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- $k = 0 \rightarrow$ original Farkas' lemma.

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with $r_1, \dots, r_k > 0$, $r_{k+1} \geq 0$.

- Using this result, we can generate **all** infeasible SDP problems, as:

(1) Generate a system like $(\mathrm{P_{ref}})$.

(2) Reformulate it.

Proof outline

- **Based on simplified facial reduction algorithm: construct the A_i' one by one.**

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- **Alternative: adapt a traditional facial reduction algorithm, the closest one is by Waki and Muramatsu.**

Computational use

- **Infeasible instances with this basic structure are very challenging for SDP solvers!**
- **Even more so, if we apply random elementary row ops and rotations.**

Papers

- **P: On the closedness of the linear image of a closed convex cone, Math of OR, 2007**
- **P: Bad semidefinite programs: they all look the same, under review.**
- **Liu-P: Exact duality in semidefinite programming based on elementary reformulations, SIOPT 2015**
- **Liu-P: Exact duals and short certificates fo infeasibility and weak infeasibility in conic linear programming, under review**

Conclusion

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- $(\mathrm{P}_{\mathrm{ref}})$ **being infeasible is almost a tautology.**

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- **Block-diagonality** of **all** dual multipliers
- Exact, simple certificate of infeasibility of a semidefinite system based on elementary reformulation.
- (P_{ref}) being infeasible is almost a tautology.
- Algorithm to systematically generate **all** infeasible SDPs.

Thank you!