Bad semidefinite programs with short proofs

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Talk at Fields Institute workshop

Dedicated to Arkadi Nemirovski's birthday, 2017

A pair of Semidefinite Programs (SDP)

$$\sup_x \; c^T x \qquad\qquad \inf_Y \; B \bullet Y$$

$$s.t. \; \sum_{i=1}^m x_i A_i \preceq B \qquad\qquad Y \succeq 0$$

$$A_i \bullet Y = c_i \; (i = 1, \ldots, m).$$

Here

- A_i, B are symmetric matrices, $c, x \in \mathbb{R}^m$.
- $A \preceq B$ means that $B - A$ is symmetric positive semidefinite (psd).
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

SDP duality

The primal-dual pair of SDPs:

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Ideal situation: $\exists \bar{x}, \exists \bar{Y} : c^T \bar{x} = B \bullet \bar{Y}$.

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But: in SDP, unlike in LP pathological phenomena occur: nonattainment, positive gaps.

This is bad, since we would like a certificate of optimality.

Pathology # 1: nonattainment in dual

Primal:

$$\begin{aligned} \sup \ & 2x_1 \\ s.t. \ & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

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Unattained $\inf = 0.$

Other pathologies

- **Positive duality gaps; positive gap and nonattainment; etc.**

Terminology

Definition:

- **The system**

$$(P_{SD}) \ \sum_{i=1}^m x_i A_i \preceq B$$

is badly behaved if $\exists c$ such that

$$\sup\{ c^T x \,|\, x \in (P_{SD}) \} < +\infty$$

but the dual program has no solution with same value (i.e. dual does not attain, or positive gap).

- **Well behaved, otherwise.**
- **We would like to understand well/badly behaved systems.**

Motivation

The systems

$$x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

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Curious similarity – of these, and about 20 others in the literature

Why all bad SDPs look the same

- **Semidefinite system:**

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- Ex: $x_1 \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{V} \preceq \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^{Z}$

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- **Matrices Z, V prove that (P_{SD}) is badly behaved.**
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- Aside: how do we prove that $Ax = b$ is infeasible? $\rightarrow$ row echelon form with $\langle 0, x \rangle = 1$.
- We will borrow ideas from the row echelon form to produce easy-to-verify certificates.

Reformulations of

$$(P_{SD})\ \sum_{i=1}^m x_i A_i \preceq B$$

are obtained by a sequence of:

- Rotate all matrices by $T = \begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix}$, M orthogonal.
- $B \leftarrow B + \sum_{i=1}^m \mu_i A_i$
- $A_i \leftarrow \sum_{j=1}^m \lambda_j A_j$ where $\lambda_i \neq 0$

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Origin: Elemantary row operations on dual.

E.g.replace $A_i \bullet Y = c_i$ by $\sum_j (\lambda_j A_j) \bullet Y = \sum_j \lambda_j c_j$.

Theorem: (P_{SD}) is badly behaved $\Leftrightarrow$ it has a reformulation:

$$(P_{SD,bad}) \ \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

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Proof that $(P_{SD,bad})$ **is badly behaved:**

x **feas. with slack** S $\Rightarrow$ **last** $n - r$ **cols of** S **are zero**

$\Rightarrow \quad x_{k+1} = \cdots = x_m = 0$

$\Rightarrow \quad \sup -x_m = 0$

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Note partitioning into

- **"Slater part" with** $x_1, \dots, x_k$ **and**
- **"Redundant part" with** $x_{k+1}, \dots, x_m$

Example: before reformulation

$$x_1 \begin{pmatrix} 54 & 46 & 50 & 4 \\ 46 & -38 & 87 & -106 \\ 50 & 87 & -60 & 296 \\ 4 & -106 & 296 & -368 \end{pmatrix} + x_2 \begin{pmatrix} 110 & 91 & 105 & -6 \\ 91 & -72 & 171 & -210 \\ 105 & 171 & -72 & 528 \\ -6 & -210 & 528 & -672 \end{pmatrix} + x_3 \begin{pmatrix} 42 & 35 & 40 & 0 \\ 35 & -28 & 67 & -82 \\ 40 & 67 & -36 & 216 \\ 0 & -82 & 216 & -272 \end{pmatrix}$$

$$+ x_4 \begin{pmatrix} 36 & 30 & 35 & -2 \\ 30 & -24 & 57 & -70 \\ 35 & 57 & -24 & 176 \\ -2 & -70 & 176 & -224 \end{pmatrix} \preceq \begin{pmatrix} 389 & 323 & 370 & -12 \\ 323 & -257 & 610 & -748 \\ 370 & 610 & -288 & 1920 \\ -12 & -748 & 1920 & -2432 \end{pmatrix}$$

Hard to tell if well or badly behaved

Example: after reformulation

$$x_1\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + x_2\left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + x_3\left(\begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & -1 \\ \hline 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{array}\right)$$

$$+x_4\left(\begin{array}{cc|cc} 0 & 0 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ \hline 3 & 2 & 4 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \preceq \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

As before: $x_3 = x_4 = 0 \Rightarrow \sup -x_4 = 0$

But: no dual solution with value 0

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Story continued

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- Proofs:
  - 1) characterize badly behaved conic LPs, 2) specialize to SDPs
  - Uses "On the closedness of linear image of a closed convex cone", P 2007, MOR
  - Results from 3-4 papers combined

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- We would like a simpler, combinatorial proof

A much simpler proof

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The bad part

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it has a "Bad reformulation" $(P_{SD,bad}) \implies$

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Proof Basic convex analysis: Gordan-Stiemke theorem.

Gordan-Stiemke theorem

Given closed convex cone K and linear subspace L

$$\mathrm{ri}\, K \cap L^{\perp} = \emptyset \;\Leftrightarrow\; (K^{*} \setminus K^{\perp}) \cap L \neq \emptyset.$$

Good condition fails $\Longrightarrow$ Bad condition holds.

Good condition $(1)\ \exists U \succ 0\ s.t.$

$$A_i \bullet \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix} = 0\ \forall i.$$

(2) If V is a linear combination of the A_i

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Good (1) fails $\Longrightarrow$ Bad holds (use Gordan-Stiemke)

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- Combinatorial type characterizations.
- Reformulations into **canonical forms** to easily recognize good and bad behavior.
- Now: **elementary proofs** , with:
  - Basic linear algebra
  - One application of Gordan-Stiemke theorem
- Also in this paper: when is the linear image of the semidefinite cone closed?
- Other uses of canonical forms:
  - "Easy" certificate of infeasibility for SDP: Liu-P, SIOPT 2015
  - "Easy" certificate of infeasibility and weak infeasibility for conic LP: Liu-P, MPA 2017

Happy birthday! and Thank you!