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ON THE RANK OF EXTREME MATRICES IN SEMIDEFINITE PROGRAMS AND THE MULTIPLICITY OF OPTIMAL EIGENVALUES

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We derive some basic results on the geometry of *semidefinite programming* (*SDP*) and *eigenvalue-optimization*, i.e., the minimization of the sum of the k largest eigenvalues of a smooth matrix-valued function.

We provide upper bounds on the rank of extreme matrices in SDPs, and the first theoretically solid explanation of a phenomenon of intrinsic interest in eigenvalue-optimization. In the spectrum of an optimal matrix, the kth and $(k + 1)$st largest eigenvalues tend to be equal and frequently have multiplicity greater than two. This clustering is intuitively plausible and has been observed as early as 1975.

When the matrix-valued function is affine, we prove that clustering *must* occur at extreme points of the set of optimal solutions, if the number of variables is sufficiently large. We also give a lower bound on the multiplicity of the critical eigenvalue. These results generalize to the case of a general matrix-valued function under appropriate conditions.

1. Introduction. A *semidefinite programming problem* (SDP) can be formulated in the form

$$\begin{aligned} &\text{Min } C \bullet X \\ &\text{s.t.} \quad X \succeq 0, \\ &A_i \bullet X = b_i \quad (i = 1, \ldots, m), \end{aligned} \tag{1.1}$$

where A_i $(i = 1, \ldots, m)$, C are symmetric matrices, b_i $(i = 1, \ldots, m)$ real numbers, the inner product of symmetric matrices is $A \bullet B = \sum_{i,j=1}^{n} a_{ij} b_{ij}$, and $X \succeq 0$ means that the matrix X is symmetric and positive semidefinite. The dual of (1.1) is

$$\begin{aligned} &\text{Max } y^T b \\ &\text{s.t.} \quad Z \succeq 0, \\ &\sum_{i=1}^{m} y_i A_i + Z = C. \end{aligned} \tag{1.2}$$

The origins of semidefinite programming can be traced back to the seventies; however it has gained tremendous popularity only in the past few years. The importance of SDP is due to several facts. It is an elegant generalization of linear programming, and to a large extent inherits its duality theory. Also, it has a wealth of applications ranging from en-

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gineering to combinatorial optimization. For an extensive historic account, we refer to Alizadeh (1995). A more recent survey is the paper of Vandenberghe and Boyd (1996).

A related problem can be described as follows. For an n by n symmetric matrix B denote by $\lambda_i(B)$ the ith largest eigenvalue of B. Let $k \in \{1, \ldots, n\}$, and define

$$f_k(B) = \sum_{i=1}^{k} \lambda_i(B). \tag{1.3}$$

Let A be a smooth function mapping from $\mathcal{R}^m$ to the space of symmetric n by n matrices. The *eigenvalue-optimization problem* is

$$\text{Min}\{f_k(A(x)) : x \in \mathcal{R}^m\}. \tag{EV_k}$$

When A is affine we call (EV_k) an *affine*; and a *general problem* otherwise. As it was shown by Nesterov and Nemirovskii (1994) and Alizadeh (1995), the affine problem can be formulated as an SDP; Alizadeh's paper is also an excellent chronicle on this special case. A recent, comprehensive survey on optimizing functions of eigenvalues was written by Lewis and Overton (1996).

The purpose of this paper is to describe several basic results on the geometry of semidefinite programs and eigenvalue-optimization. First we derive upper bounds on the rank of extreme matrices in SDPs. These bounds are similar to the well-known bounds on the number of nonzeros in extreme solutions of linear programs.

Next, we study a phenomenon of intrinsic interest in eigenvalue-optimization. At optimal solutions of (EV_k) the eigenvalues of the optimal matrix tend to coalesce; if the minimum is achieved at x^*, then frequently $\lambda_k(A(x^*)) = \lambda_{k+1}(A(x^*))$, and $\lambda_k(A(x^*))$ can have multiplicity larger than two.

The clustering phenomenon plays a central role in eigenvalue-optimization. The function f_k is differentiable at B if and only if $\lambda_k(B) > \lambda_{k+1}(B)$. If this condition fails to hold, then the dimension of the subdifferential of f_k at B grows quadratically with the multiplicity of $\lambda_k(B)$. Furthermore, if f_k is nonsmooth at $A(x^*)$ then generally the composite function $f_k \circ A$ is also nonsmooth at x^*. For the characterization of the subdifferential of f_k and $f_k \circ A$ we refer the reader to Overton and Womersley (1993) and Hiriart-Urruty and Ye (1995). A more general treatment on computing subdifferentials of functions of eigenvalues can be found in Lewis (1996).

Therefore, clustering frequently causes the nondifferentiability of the objective function at a solution point, making (EV_k) a ''model problem'' in nonsmooth optimization. The reason for clustering is intuitively clear: the optimization objective (''pushing down'' the sum of the k largest eigenvalues) makes the eigenvalues coalesce around $\lambda_k(A(x^*))$.

We provide a theoretically sound explanation of the clustering phenomenon. For the affine problem we prove that at a point x^*, which is an extreme point of the set of optimal solutions, if $m > k(n-k)$ then $\lambda_k(A(x^*)) = \lambda_{k+1}(A(x^*))$ must hold, and there is a lower bound on the multiplicity of $\lambda_k(A(x^*))$; this bound is an increasing function of m. For the general problem we show that if x^* is an optimal solution of (EV_k), then $A(x^*)$ minimizes f_k on the tangent space of the set $\{A(x) \mid x \in \mathcal{R}^m\}$ at $A(x^*)$. If the restriction of f_k to the tangent space is strictly convex at $A(x^*)$, then the same results hold as in the affine case.

The rest of the paper is organized as follows. In the remainder of this section we introduce the necessary notation and review preliminaries. In §2 we derive the bounds on the rank of extreme matrices in SDPs. In §3 we study a simple semidefinite program whose optimum is $f_k(B)$ for a fixed symmetric matrix B. We derive a closed form expression for the optimal solution. Section 4 proves our main result, Theorem 4.3 on the multiplicity of optimal eigenvalues in the affine case. The proof utilizes the results of the previous two sections. In particular, the *upper* bounds on the ranks of the slack matrices in the corresponding SDP translate into a *lower* bound on the multiplicity of the kth largest eigenvalue. Section 5 treats the general case. In §6 we outline how similar arguments can be applied to a similar problem: minimizing the sum of the k eigenvalues which are largest in absolute value.

Notation and preliminaries.

Linear algebra. $\mathcal{S}^n$ denotes the set of n by n symmetric matrices, $\mathcal{S}^n_+$ the set of n by n symmetric positive semidefinite matrices. For $A, B \in \mathcal{S}^n$, $A \succeq B$ $[A \succ B]$ means that $A - B$ is positive semidefinite [positive definite]. We denote by $t(n)$ the nth triangular number $\frac{1}{2}n(n+1)$.

For $B \in \mathcal{S}^n$ the ith largest eigenvalue of B is denoted by $\lambda_i(B)$. Also, mult $(\lambda_i(B))$ denotes the multiplicity of $\lambda_i(B)$, that is the maximal $p \geq 1$ such that

$$\lambda_j(B) = \cdots = \lambda_i(B) = \cdots = \lambda_{j+p-1}(B)$$

for some j such that $j \leq i \leq j + p - 1$.

For a matrix $A = (a_{ij})^n_{i,j=1}$ we denote by diag A the vector $(a_{11}, \ldots, a_{nn})^T$. For $v \in \mathcal{R}^n$ we denote by Diag v the diagonal matrix with diagonal elements $v_1, \cdots, v_n$.

We denote by e the vector of all ones, by e^i the ith unit vector.

Convex analysis. Let S be a closed convex set. A convex subset F of S is called a *face* of S if $x \in F$, $y, z \in S$, $x = \frac{1}{2}(y + z)$ implies that y and z must both be in F. A *vertex* or an *extreme point* of S is a face consisting of a single element.

The vectors $v^1, \ldots, v^p \in \mathcal{R}^n$ are *affinely independent*, if

$$\sum_{i=1}^{p} \mu_i v^i = 0, \qquad \sum_{i=1}^{p} \mu_i = 0$$

implies $\mu_1 = \cdots = \mu_p = 0$. We say that the matrices $A_1, \ldots, A_p$ are linearly (affinely) independent, if the vectors formed of $A_1, \ldots, A_p$ by stacking their columns are linearly (affinely) independent.

The *dimension* of the convex set S is

$$\dim S = \max\{p \mid v^1, \ldots, v^p \in S \text{ are affinely independent}\} - 1.$$

The standard reference for convex analysis is Rockafellar (1970).

Semidefinite programming. For the ease of reference we state the fundamental theorem of SDP-duality. It is a special case of duality for linear programs with cone-constraints. For a proof see, e.g., Wolkowicz (1981). Also, a general formulation of SDP duality theory was given in Shapiro (1985).

THEOREM 1.1. *Assume that both* (1.1) *and* (1.2) *have feasible solutions. Then the following results hold.*

(1) *If* (1.1) *has a feasible* $X \succ 0$, *then the optimal values of* (1.1) *and* (1.2) *are equal, and* (1.2) *attains its optimal value.*

(2) *If* (1.2) *has a feasible* (y, Z) *with* $Z \succ 0$, *then the optimal values of* (1.1) *and* (1.2) *are equal, and* (1.1) *attains its optimal value.*

2. Bounding the rank of extreme matrices in semidefinite programs.

THEOREM 2.1. *The following results hold*:

1. *Suppose that* $X \in F$, *where* F *is a face of the feasible set of* (1.1). *Let* $d = \dim F$, $r = \operatorname{rank} X$. *Then*

$$t(r) \le m + d. \tag{2.1}$$

2. *Suppose that* $(y, Z) \in G$, *where* G *is a face of the feasible set of* (1.2). *Let* $d = \dim G$, $s = \operatorname{rank} Z$. *Then*

$$t(s) \le t(n) - m + d. \tag{2.2}$$

PROOF OF 1. Since $\operatorname{rank} X = r$, we can write

$$X = Q\Lambda Q^T$$

where $Q \in \mathcal{R}^{n\times r}$, $\Lambda \in \mathcal{S}^r$, $\Lambda \succ 0$. Since

$$b_i = A_i \bullet X = A_i \bullet Q\Lambda Q^T = Q^T A_i Q \bullet \Lambda \quad (i = 1, \dots, m) \tag{2.3}$$

we have

$$Q^T A_i Q \bullet \Lambda = b_i \quad (i = 1, \dots, m). \tag{2.4}$$

To obtain a contradiction, suppose that $t(r) > m + d$. The system (2.4) is determined by m equations, and $\dim \mathcal{S}^r = t(r)$. Hence there exist $\Delta_1, \dots, \Delta_{d+1} \in S^r$ linearly independent matrices satisfying

$$Q^T A_i Q \bullet \Delta_j = 0 \quad (i = 1, \dots, m; j = 1, \dots, d + 1). \tag{2.5}$$

Since Λ is positive definite, there exists $\epsilon > 0$ such that

$$\Lambda \pm \epsilon\Delta_j \succeq 0 \quad (j = 1, \dots, d + 1).$$

Define

$$\begin{aligned} \Lambda_{j,1} &= \Lambda + \epsilon\Delta_j, & \Lambda_{j,2} &= \Lambda - \epsilon\Delta_j, \\ X_{j,1} &= Q\Lambda_{j,1}Q^T, & X_{j,2} &= Q\Lambda_{j,2}Q^T. \end{aligned} \tag{2.6}$$

The matrices $X_{j,1}$ and $X_{j,2}$ are feasible for (1.1), and $X = \frac{1}{2}(X_{j,1} + X_{j,2})$. Since F is a face of the feasible set, we conclude that $X_{j,1}$ and $X_{j,2}$ are in F for $j = 1, \dots, d + 1$.

On the other hand, as $\Delta_1, \dots, \Delta_{d+1}$ are linearly independent, the matrices $\Lambda, \Lambda_{1,1}, \dots, \Lambda_{d+1,1}$ and therefore also $X, X_{1,1}, \dots, X_{d+1,1}$ are affinely independent. As the latter are in F, we obtain $\dim F \ge d + 1$, a contradiction.

PROOF OF 2. Denote by Φ the projection of the feasible set of the problem (1.2) to the space of the Z variables,

$$\Phi = \left\{ Z \succeq 0 \,|\, Z = C - \sum_{i=1}^{m} y_i A_i \text{ for some } y \in \mathcal{R}^m \right\}.$$

For $Z \in \Phi$ denote by $y(Z)$ the vector in $\mathcal{R}^m$ satisfying $Z = C - \sum_{i=1}^m y(Z)_i A_i$. Since the matrices A_i are linearly independent, $y(Z)$ is well defined. Hence the feasible set of the problem (1.2) can be written as

$$\Psi = \{(y(Z), Z) | Z \in \Phi\}.$$

The points in Φ and Ψ are in one-to-one correspondence. Moreover, it is easy to see that the mapping assigning $y(Z)$ to Z is affine (that is, $y(\mu Z_1 + (1-\mu)Z_2) = \mu y(Z_1) + (1-\mu)y(Z_2)$, for a μ real number, when $Z_1, Z_2, \mu Z_1 + (1-\mu)Z_2 \in \Phi$).

Hence also the *faces* of Φ and Ψ are in one-to-one correspondence. Namely, F is a face of Φ if and only if $G = \{(y(Z), Z) | Z \in F\}$ is a face of Ψ, and for these faces $\dim F = \dim G$.

Therefore, it is enough to establish a bound on the rank of a matrix $Z \in F$, where F is a face of Φ, $\dim F = d$. We can write Φ as

$$\Phi = \{Z \succeq 0 | A'_j \bullet Z = b'_j \ (j = 1, \ldots, t(n) - m)\}$$

for symmetric matrices A'_j and real scalars b'_j $(j = 1, \ldots, t(n) - m)$. Hence the desired result follows from the bound (2.1). □

In a semidefinite program, we may have several matrix variables constrained to be positive semidefinite, as well as a vector of unconstrained real variables. We can prove the following result.

THEOREM 2.2. *Consider a semidefinite program with feasible set*

$$X_1 \succeq 0, \ldots, X_p \succeq 0, \quad y \in \mathcal{R}^q,$$

$$\sum_{j=1}^p A_{ij} \bullet X_j = b_i \quad (i = 1, \ldots, m_1),$$

$$\sum_{j=1}^p a_{ij} X_j + \sum_{j=1}^q y_j D_{ij} = B_i \quad (i = 1, \ldots, m_2),$$

where A_{ij}, D_{ij} and B_i are symmetric matrices, and a_{ij}, b_i are scalars. Denote the order of B_i by n_i $(i = 1, \ldots, m_2)$ and let

$$m = m_1 + \sum_{i=1}^{m_2} t(n_i)$$

that is, m is the total number of equality constraints, if a constraint with a symmetric matrix right-hand side in $\mathcal{S}^n$ is counted as $t(n)$ constraints. Suppose that $(X_1, \ldots, X_p, y) \in G$, where G is a face of the feasible set. Let $d = \dim G$, $r_j = \operatorname{rank} X_j$ $(j = 1, \ldots, p)$. Then

$$\sum_{j=1}^p t(r_j) \leq m - q + d. \tag{2.7}$$

OUTLINE OF PROOF. Let $m' = m - q$. As in the proof of the second part of Theorem 2.1, we can show that the projection of the feasible set onto the space of the X_j variables is of the form

$$\Phi = \left\{ X_1 \succeq 0, \dots, X_p \succeq 0, \sum_{j=1}^{p} A'_{ij} \bullet X_j = b'_i \ (i = 1, \dots, m') \right\} \tag{2.8}$$

for symmetric matrices A'_{ij} and reals b'_i $(i = 1, \dots, m'; j = 1, \dots, p)$, and $(X_1, \dots, X_p)$ is in F, where F is a face of Φ of dimension d. Since rank $X_j = r_j$, we can write

$$X_j = Q_j \Lambda_j Q_j^T \tag{2.9}$$

where $Q_j \in \mathcal{R}^{n \times r_j}$, $\Lambda_j \in \mathcal{S}^{r_j}$, $\Lambda_j \succ 0$ $(j = 1, \dots, p)$. Similarly as in the proof of Theorem 2.1 we obtain

$$\sum_{j=1}^{p} Q_j^T A'_{ij} Q_j \bullet \Lambda_j = b_i \quad (i = 1, \dots, p). \tag{2.10}$$

The system (2.10) has m' constraints and $\dim(\mathcal{S}^{r_1} \times \cdots \mathcal{S}^{r_p}) = \sum_{j=1}^{p} t(r_j)$. The proof can be completed analogously to the proof of the first part of Theorem 2.1. □

REMARK 2.3. It is worth noting, how the bound on the number of nonzeros in extreme solutions of linear programs can be recovered from the above results. Consider an LP with feasible set

$$\Phi = \{x \in \mathcal{R}^n \,|\, x \geq 0, Ax = b\} \tag{2.11}$$

where A has m linearly independent rows. This set can be written as the feasible set of an SDP in two different ways. We can treat the variable $x \in \mathcal{R}^n$ as the direct product of n positive semidefinite matrices of order 1. Then if the point x is in a face of dimension d of Φ, and x has r nonzero components, the well-known inequality $r \leq m + d$ follows from the bound (2.7).

Alternatively, Φ is the set of diagonals of matrices in

$$\Phi' = \{X \in \mathcal{S}^n_+ \,|\, (\text{Diag } a^i) \bullet X = b_i \ (i = 1, \dots, m), X_{ij} = 0 \ (i \neq j)\}$$

where $a^1, \dots, a^m$ are the rows of A. There is a trivial correspondence between the faces of Φ and Φ'. Hence, if X is contained in a face of dimension d of Φ', and $r = \text{rank } X$, then (2.1) yields

$$t(r) \leq (m + t(n-1)) + d$$

which does *not* imply $r \leq m + d$.

REMARK 2.4. The first proof of Theorem 2.1 was given in Pataki (1994) by using a more general argument. The faces of $\mathcal{S}^n_+$ are in one-to-one correspondence with the subspaces of $\mathcal{R}^n$. It was shown by Barker and Carlson (1975) that a convex subset of $\mathcal{S}^n_+$ is a face, if and only if it is of the form

$$F(L) = \{X \in \mathcal{S}^n_+ \,|\, \text{the rangespace of } X \text{ is contained in } L\}$$

where L is a subspace of $\mathcal{R}^n$. Let $r = \dim L$. Then the dimension of $F(L)$ is $t(r)$, and the rank of matrices in $F(L)$ is at most r.

Now, consider a different proof of Theorem 2.1. Let us denote by

$$H = \{X : A_i \bullet X = b_i \text{ for } i = 1, \ldots, m\}.$$

Then the feasible set of program (1.1) is given by $\mathcal{S}_+^n \cap H$. By a theorem of Dubins (see, e.g., Stoer and Witzgall 1970, p. 116) a subset of the intersection of two convex sets is a face iff it is the the intersection of two faces. Therefore F is a face of $\mathcal{S}_+^n \cap H$, if and only if

$$F = F(L) \cap H \tag{2.12}$$

for an L subspace of $\mathcal{R}^n$. As shown in Pataki (1994) if $F(L)$ is the *minimal* face satisfying (2.12), then

$$\dim F \geq \dim F(L) - m \Leftrightarrow \dim F(L) \leq \dim F + m. \tag{2.13}$$

Then (2.13) is equivalent to (2.1). The technique used in the current proof of Theorem 2.1 are similar to the techniques used in the proof of Theorem 5 in Alizadeh, Haeberly, and Overton (1997).

REMARK 2.5. Independently, Ramana and Goldman (1995) derived several results on the geometry of the feasible sets of SDPs (such as a characterization of faces).

3. A simple semidefinite characterization of $f_k(B)$. In this section we shall develop the second part of the theory necessary to prove eigenvalue-clustering in the optimal solutions of (EV_k). Let B be a symmetric matrix of order n, and consider the SDP

$$\begin{aligned} &\text{Min } kz + I \bullet V \\ &\text{s.t.} \quad V, W \succeq 0, \\ &zI + V - W = B. \end{aligned} \tag{3.14}$$

We shall give an explicit expression for the optimal solutions of (3.14) and prove that its optimal value is $f_k(B) = \lambda_1(B) + \cdots + \lambda_k(B)$.

We begin with a brief survey of previous results related to ours. The classical characterization

$$\begin{aligned} f_k(B) = &\text{ Max } B \bullet XX^T \\ &\text{s.t.} \quad X \in \mathcal{R}^{n\times k}, \\ &X^TX = I, \end{aligned} \tag{3.15}$$

is due to Ky Fan (1949). Showing the inequality $\leq$ is easy, by choosing the columns of X as a set of eigenvectors corresponding to the k largest eigenvalues of B.

A more recent characterization that was obtained independently by Overton and Womersley (1992, 1993) and Hiriart-Urruty and Ye (1995) states

$$\begin{aligned} f_k(B) = &\text{ Max} \quad B \bullet U \\ &\text{s.t.} \quad I \succeq U \succeq 0, \\ &I \bullet U = k. \end{aligned} \tag{3.16}$$

This result can be regarded as a "continuous Ky Fan-theorem," since these references show

$$\{U \in \mathcal{S}^n \mid I \succeq U \succeq 0, I \bullet U = k\} = \operatorname{conv}\{XX^T \mid X \in \mathcal{R}^{n \times k}, X^TX = I\} \tag{3.17}$$

and (3.17) can be used to give a simple and elegant proof of Ky Fan's original characterization (see, e.g., Overton and Womersley 1992). Furthermore, in Overton and Womersley (1993) and Hiriart-Urruty and Ye (1995) an explicit expression for the set of optimal solutions of (3.16) is given.

Problem (3.14) was studied independently by Nesterov and Nemirovskii (1994, p. 238) and Alizadeh (1995). They showed that its optimal value is $f_k(B)$. The proof of Nesterov and Nemirovskii is direct. Alizadeh proved his result by showing that (3.16) and (3.14) are dual semidefinite programs with equal optimal value (the latter has a strictly interior feasible solution).

For our purposes, we need not only the optimal value of (3.14), but also an explicit expression for the optimal z, V and W. We proceed as follows: first we determine the optimal solutions for a diagonal B, with the restriction that V and W be also diagonal. Next we drop the diagonality restriction on V and W, while keeping it on B. Finally, we determine the optimal solutions of (3.14).

LEMMA 3.1. *Let $\lambda \in \mathcal{R}^n$, $\lambda_1 \geq \cdots \geq \lambda_n$. Consider the LP*

$$\begin{aligned} &\text{Min } kz + e^T v \\ &\text{s.t.} \quad v, w \geq 0, \\ &ze + v - w = \lambda. \end{aligned} \tag{3.18}$$

The optimal value of (3.18) *is* $\sum_{i=1}^k \lambda_i$ *and* (z^*, v^*, w^*) *is an optimal solution if and only if*

$$\lambda_{k+1} \leq z^* \leq \lambda_k, \tag{3.19}$$

$$\begin{aligned} v^* &= (\lambda_1 - z^*, \ldots, \lambda_k - z^*, \quad 0, \quad \ldots, \quad 0 \quad)^T, \\ w^* &= (0, \quad \ldots, \quad 0, \quad z^* - \lambda_{k+1}, \ldots, z^* - \lambda_n)^T. \end{aligned} \tag{3.20}$$

PROOF. Let (z, v, w) be feasible for (3.18). Then

$$\begin{aligned} kz + e^T v &\geq (ze + v)^T(e^1 + \cdots + e^k) \\ &= (\lambda + w)^T(e^1 + \cdots + e^k) \\ &\geq \lambda_1 + \cdots + \lambda_k. \end{aligned}$$

The first inequality is tight if and only if $v_{k+1} = \cdots = v_n = 0$. The second inequality is tight if and only if $w_1 = \cdots = w_k = 0$, thus the choice of the optimal (z^*, v^*, w^*) follows. □

LEMMA 3.2. *Let λ be as in Lemma* 3.1 *and* $\Lambda = \operatorname{Diag} \lambda$. *Consider the SDP*

$$\begin{aligned} &\operatorname{Min}\; kz + I \bullet V \\ &\text{s.t.} \quad V, W \succeq 0, \\ &zI + V - W = \Lambda. \end{aligned} \tag{3.21}$$

The optimal value of (3.21) *is* $\Sigma_{i=1}^{k} \lambda_i$, *and* (z^*, V^*, W^*) *are optimal if and only if* $V^* = \operatorname{Diag} v^*$, $W^* = \operatorname{Diag} w^*$, *and* (z^*, v^*, w^*) *are chosen as in* (3.19) *and* (3.20).

PROOF. Replacing the constraints $V \succeq 0$, $W \succeq 0$ with diag $V \geq 0$, diag $W \geq 0$ yields (3.18), a relaxation of (3.21). Thus (3.21) has optimal value at least $\Sigma_{i=1}^{k} \lambda_i$. This value is attained if both V^* and W^* are diagonal.

We must also show that *arbitrary* optimal V^* and W^* matrices are diagonal. Let (z^*, V^*, W^*) be an optimal solution of (3.21). Therefore $(z^*, \operatorname{diag} V^*, \operatorname{diag} W^*)$ must be an optimal solution of (3.18), hence by Lemma 3.1 they are chosen according to (3.19) and (3.20). Partition the matrices V^* and W^* as

$$V^* = \begin{bmatrix} V_{11}^* & V_{12}^* \\ V_{21}^* & V_{22}^* \end{bmatrix}, \qquad W^* = \begin{bmatrix} W_{11}^* & W_{12}^* \\ W_{21}^* & W_{22}^* \end{bmatrix}, \tag{3.22}$$

where the diagonal blocks are k by k and $n - k$ by $n - k$, respectively. Since the diagonal elements of V_{22}^* are 0, and $V^* \succeq 0$, we must have $V_{22}^* = 0$ and $V_{12}^* = 0$. Since the diagonal elements of W_{11}^* are 0, and $W^* \succeq 0$, $W_{11}^* = 0$ and $W_{12}^* = 0$. Since also $z^*I + V^* - W^* = \Lambda$ holds, V^* and W^* must be diagonal. □

THEOREM 3.3. *Consider the SDP* (3.14). *Write* $B = Q\Lambda Q^T$ *with* Q *being an* n *by* n *orthonormal matrix*, $\Lambda = \operatorname{diag} \lambda$, $\lambda_1 \geq \cdots \geq \lambda_n$. *The optimal value of* (3.14) *is* $\Sigma_{i=1}^{k} \lambda_i$, *and* (z^*, V^*, W^*) *are optimal if and only if*

$$\lambda_{k+1} \leq z^* \leq \lambda_k, \tag{3.23}$$

$$\begin{aligned} V^* &= Q(\operatorname{Diag} v^*)Q^T, \\ W^* &= Q(\operatorname{Diag} w^*)Q^T, \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} v^* &= (\lambda_1 - z^*, \ldots, \lambda_k - z^*, \quad 0, \quad \ldots, \quad 0 \quad)^T, \\ w^* &= (0, \quad \ldots, \quad 0, \quad z^* - \lambda_{k+1}, \ldots, z^* - \lambda_n)^T. \end{aligned} \tag{3.25}$$

PROOF. Given the above decomposition of B we can rescale (3.14) to get (3.21) without changing its objective value. The correspondence between the solutions of the rescaled and the original problem is: (z^*, V^*, W^*) is an optimal solution to the rescaled problem if and only if (z^*, QV^*Q^T, QW^*Q^T) is an optimal solution to (3.14). Our theorem then follows from Lemma 3.2. □

Finally we remark that the optimal solutions of (3.14) do not depend on the choice of B's eigenvectors. Suppose that the distinct eigenvalues of B are

$$\lambda_{i_1}(B), \ldots, \lambda_{i_r}(B)$$

in descending order and $\lambda_k(B) = \lambda_{i_s}(B)$. Then the distinct eigenvalues of V^* and W^* in Theorem 3.3 are

$$\lambda_{i_1}(B) - z^*, \ldots, \lambda_{i_{s-1}}(B) - z^* \quad \text{and} \quad z^* - \lambda_{i_{s+1}}(B), \ldots, z^* - \lambda_{i_{s-1}}(B),$$

respectively. Therefore, choosing different eigenvectors of B to represent the eigenspace corresponding to $\lambda_{i_j}(B)$ $(j = 1, \ldots, r)$ does not change V^* and W^*.

In the following we shall denote

$$\Omega_k(B) = \{(z, V, W) \mid (z, V, W) \text{ is an optimal solution of } (3.14)\}.$$

This set is a line segment if $\lambda_k(B) > \lambda_{k+1}(B)$ and a singleton otherwise. Let $(z, V, W) \in \Omega_k(B)$ with $z = \frac{1}{2}(\lambda_k(\mathrm{B}) + \lambda_{k+1}(\mathrm{B}))$. Then from Theorem 3.3 we obtain

$$\operatorname{rank} V + \operatorname{rank} W + \operatorname{mult}(\lambda_k(B)) = n.$$

4. Multiplicity of optimal eigenvalues: The affine case. In this section we present our main results about the multiplicity of optimal eigenvalues in the affine case.

Let $A_0, A_1, \ldots, A_m$ be symmetric matrices, that we assume to be linearly independent. Let $A(x) = A_0 + \sum_{i=1}^m x_i A_i$. Substituting $A(x)$ for B in (3.14) yields the SDP-formulation for the affine problem

$$\begin{aligned} &\operatorname*{Min}_{x,z,V,W} \; kz + I \bullet V \\ &\text{s.t.} \quad V, W \succeq 0, \\ &zI + V - W = A(x). \end{aligned} \tag{4.26}$$

This formulation was discovered independently by Nesterov and Nemirovskii (1994) and Alizadeh (1995). The results of the previous section show that the optimal solutions of program (4.26) are determined by the optimal solutions of (EV_k): the only possible degree of freedom we have is choosing z^*.

Denote the optimal value of (EV_k) by f_k^* and the set of optimal solutions by $\mathcal{O}$, i.e.

$$\mathcal{O} = \{x \in \mathcal{R}^m \mid f_k(A(x)) = f_k^*\}. \tag{4.27}$$

The set $\mathcal{O}$ is a level set of the function $f_k \circ A$ (in its definition we can write $f_k(A(x)) \leq f_k^*$) hence it is convex and closed.

To exclude trivial cases, from now on we assume $m \geq 1$ and $k < n$ (when $k = n$, (EV_k) is a linear program).

LEMMA 4.1. *The set $\mathcal{O}$ does not contain a line, i.e., if $y \in \mathcal{R}^m$, $x + \lambda y \in \mathcal{O}$ for all $\lambda \in \mathcal{R}$ then $y = 0$ must hold.*

PROOF. Since $m \geq 1$, and the A_i matrices are linearly independent, it suffices to show that the level set

$$\mathcal{C} = \{Z \in \mathcal{S}^n \mid f_k(Z) \leq f_k^*\}$$

does not contain a line. Suppose that Y is a symmetric matrix such that $Z + \lambda Y \in \mathcal{C}$ for all $\lambda \in \mathcal{R}$. Since $f_k(Z + \lambda Y)$ can be determined from Ky Fan's formulation (3.15) it follows that $Y \bullet XX^T = (-Y) \bullet XX^T = 0$ must hold for all X n by k matrices satisfying

$X^TX = I$. This in turn implies $f_k(Y) = f_k(-Y) = 0$. Since $k < n$, we conclude $Y = 0$ as needed. □

Since $\mathcal{O}$ does not contain a line, it has at least one extreme point (Rockafellar 1970, Corollary 18.5.3).

LEMMA 4.2. *Let x^* be an extreme point of $\mathcal{O}$. Then the set*

$$F^* = \{x^*\} \times \Omega_k(A(x^*))$$

is a face of the feasible set of (4.26).

PROOF. Denote the feasible set of (4.26) by Φ, and the set of its optimal solutions by F. We have

$$\begin{aligned} F &= \{(x, z, V, W) \in \Phi \,|\, kz + I \bullet V = f_k^*\} \\ &= \{\{x\} \times \Omega_k(A(x)) \,|\, x \in \mathcal{O}\}. \end{aligned}$$

Since x^* is an extreme point of $\mathcal{O}$, F^* must be a face of F. As F is a face of Φ, our claim follows. □

We recall the definition of the function t, and introduce the function τ

$$\begin{aligned} t(i) &= i(i+1)/2, \\ \tau(l, r, s) &= \max\{i + j : t(i) + t(j) \le l,\ i \le r,\ j \le s\}. \end{aligned} \tag{4.28}$$

THEOREM 4.3. *Let x^* be an extreme point of $\mathcal{O}$, and assume $m > k(n-k)$. Then*

$$\lambda_k(A(x^*)) = \lambda_{k+1}(A(x^*)), \quad \textit{and} \tag{4.29}$$

$$\operatorname{mult}(\lambda_k(A(x^*))) \ge n - \tau(t(n) - m - 1, k - 1, n - k - 1). \tag{4.30}$$

PROOF. Define F^* as in Lemma 4.2, let $d = \dim F^*$, and denote $\lambda_i = \lambda_i(A(x^*))$ $(i = 1, \ldots, n)$. Clearly,

$$d = \begin{cases} 1 & \text{if } \lambda_k > \lambda_{k+1}, \\ 0 & \text{if } \lambda_k = \lambda_{k+1}. \end{cases} \tag{4.31}$$

Let $(x^*, z, V, W) \in F^*$, with $z = \frac{1}{2}(\lambda_k + \lambda_{k+1})$, and let $i = \operatorname{rank} V$, $j = \operatorname{rank} W$. Then clearly

$$i \le k \quad \text{and} \quad j \le n - k.$$

Moreover, in program (4.26) the number of equality constraints is $t(n)$, and the number of unconstrained variables is $m + 1$. Hence Theorem 2.2 implies

$$t(i) + t(j) \le (t(n) - m - 1) + d. \tag{4.32}$$

Since $m > k(n-k)$, we get $t(k) + t(n-k) > (t(n) - m - 1) + 1$, hence $i = k$ and $j = n - k$ cannot both hold. Therefore

$$\lambda_k(A(x^*)) = \lambda_{k+1}(A(x^*))$$

follows. To prove (4.30) note that

$$i + j + \operatorname{mult}(\lambda_k(A(x^*))) = n$$

and

$$d = 0, \qquad i \le k - 1, \qquad j \le n - k - 1.$$

By (4.32) we get

$$i + j \le \tau(t(n) - m - 1, k - 1, n - k - 1) \Rightarrow$$

$$\operatorname{mult}(\lambda_k(A(x^*))) \ge n - \tau(t(n) - m - 1, k - 1, n - k - 1)$$

as required. □

Next, we study the consequences of Theorem 4.3 on the nonsmoothness of the objective function of (EV_k) at an optimal solution. Ky Fan's formula (3.15) and the SDP (3.16) provide a characterization of f_k as the pointwise maximum of (infinitely many) linear functions. Hence the subdifferential of f_k at B is given as (Rockafellar 1970, p. 214)

$$\partial f_k(B) = \operatorname{conv}\{XX^T \mid X \in \mathcal{R}^{n\times k}, X^TX = I, B \bullet XX^T = f_k(B)\} \tag{4.33}$$

$$= \{U \in \mathcal{S}^n \mid I \succeq U \succeq 0, I \bullet U = k, B \bullet U = f_k(B)\} \tag{4.34}$$

(the convex hull operation is not needed in (4.34) as the set of optimal solutions of (3.16) is convex). It can be shown that $X \in \mathcal{R}^{n\times k}$ with $X^TX = I$ satisfies $B \bullet XX^T = f_k(B)$ if and only if its columns are eigenvectors of B corresponding to its k largest eigenvalues. The "if" part is obvious; the "only if" part can be proved by using the result of Overton and Womersley (1993) and Hiriart-Urruty and Ye (1995) characterizing the set in (4.34). Hence f_k is not differentiable at B if and only if $\lambda_k(B) = \lambda_{k+1}(B)$. If this condition is satisfied, then $\dim \partial f_k(B) = t(\operatorname{mult}(\lambda_k(B)))$ holds (see the above references).

Hence Theorem 4.3 proves that the function f_k is nonsmooth at $A(x^*)$, moreover, it gives a lower bound on the dimension of $\partial f_k(A(x^*))$.

The subdifferential of the *composite function* $f_k \circ A$ at x^* can be computed as (see the above references)

$$\partial(f_k \circ A)(x^*) = \{(A_1 \bullet U, \ldots, A_m \bullet U)^T \mid U \in \partial f_k(A(x^*))\}.$$

Therefore $f_k \circ A$ will *generally* be nonsmooth at x^* when f_k is nonsmooth at $A(x^*)$. This is not always true; a trivial counterexample is when $A(x) \equiv I$.

REMARK 4.4. For a matrix $B \in \mathcal{S}^n$ we can define the *interior* and *exterior* multiplicities of the eigenvalue $\lambda_k(B)$ as (see Cullum, Donath, and Wolfe 1995)

$$\operatorname{mult}_{\mathrm{int}}(\lambda_k(B)) = \max\{j \mid \lambda_{k-j}(B) = \lambda_k(B)\}$$

$$\operatorname{mult}_{\mathrm{ext}}(\lambda_k(B)) = \max\{j \mid \lambda_{k+j}(B) = \lambda_k(B)\}.$$

Note that Theorem 4.3 proves $\operatorname{mult}_{\mathrm{ext}}(\lambda_k(A(x^*)) \ge 1$, and gives a lower bound on the *sum* of the interior and exterior multiplicities, but not on either of them. The reason is that in the proof we can guarantee an upper bound on the *sum* of the ranks of the slack matrices, but not on any of them. (Of course, if $k \le n/2$ and m is large enough to guarantee $\operatorname{mult}(\lambda_k(A(x^*))) \ge k + l$ for some $l \ge 1$, then the exterior multiplicity of $\lambda_k(A(x^*))$ will have to be at least l.)

For the same reason, to prove $\lambda_k(A(x^*)) = \lambda_{k+1}(A(x^*))$ we must consider an extreme point optimal solution in the x-space; the proof does not work by looking at an extreme point optimal solution of the SDP-formulation (4.26). If (x^*, z, V, W) is such a solution, then we must have $(z, V, W) \in \Omega_k(A(x^*))$ with z being equal to either $\lambda_k(A(x^*))$ or $\lambda_{k+1}(A(x^*))$. Then the upper bound on rank V + rank W translates into a lower bound on $\operatorname{mult}_{\text{int}}(\lambda_k(A(x^*))) + \operatorname{mult}_{\text{ext}}(\lambda_{k+1}(A(x^*)))$, but does not prove $\lambda_k(A(x^*)) = \lambda_{k+1}(A(x^*))$, which is necessary to show the nonsmoothness of f_k at $A(x^*)$.

REMARK 4.5. Unfortunately, the bound on $\operatorname{mult}(\lambda_k(A(x^*)))$ in (4.30) is not a simple function of n, m, and k. We can calculate an explicit expression as follows. For a positive integer q and a real number l define

$$h_q(l) = \max\{\tfrac{p}{q} \mid p \text{ integer}, t(\tfrac{p}{q}) \le l\}$$

$$= \max\{\tfrac{p}{q} \mid p \text{ integer}, \tfrac{p}{q} \le \sqrt{2l + 0.25} - 0.5\}\ .$$

First notice that if $i + j$ is fixed, then $t(i) + t(j)$ is minimal, when $|i - j| \le 1$. Therefore

$$\tau(l) := \tau(l, +\infty, +\infty) = \max\{i + j : t(i) + t(j) \le l\} \tag{4.35}$$

$$= \max\{p \text{ integer} : 2t(\tfrac{p}{2}) \le l\} \tag{4.36}$$

$$= 2h_2(l/2)$$

where the equality of (4.35) and (4.36) follows, since l is integer, and $t(\lfloor \frac{p}{2} \rfloor) + t(\lceil \frac{p}{2} \rceil)$ and $2t(\frac{p}{2})$ differ by at most 1/4. Also, if $r \le s$, $\tau(l) \le r + s$ then the maximum of $i + j$ in the definition of $\tau(l, r, s)$ is attained if the difference of i and j is minimal. Therefore

$$\tau(l, r, s) = \begin{cases} r + h_1(l - t(r)), & \text{if } r < \lfloor h_2(l/2) \rfloor \\ 2h_2(l/2), & \text{otherwise.} \end{cases} \tag{4.37}$$

Since the bound on $\operatorname{mult}(\lambda_k(A(x^*)))$ is the same if we replace k by $n - k$ we may assume $k \le n/2$. Also, $m > k(n - k)$ implies $\lambda_k(A(x^*)) = \lambda_{k+1}(A(x^*))$, hence we may assume that m is large enough, so that

$$\tau(t(n) - m - 1) \le n - 2.$$

Therefore we can substitute $l = t(n) - m - 1$, $r = k - 1$, $s = n - k - 1$ into (4.37) to compute the bound in (4.30) explicitly. For $n = 100$, the graph of Figure 1 plots the bound on the multiplicity of $\operatorname{mult}(\lambda_k(A(x^*)))$ as a function of m for $k = 1$ and $k = 50$.

REMARK 4.6. $\operatorname{mult}(\lambda_k(A(x^*)))$ can be large, even if m is small, as the following example shows. Let $m = 1$, $A_0 = I$, and A_1 an arbitrary n by n symmetric matrix that has a k by k principal minor equal to a 0 matrix. Since $A(x)$ has I_k as a principal minor regardless of the choice of x, $f_k(A(x)) \ge k$ for arbitrary x (e.g., by Ky Fan's theorem). $x^* = 0$ is optimal, and the multiplicity of $\lambda_k(A(x^*))$ is n.

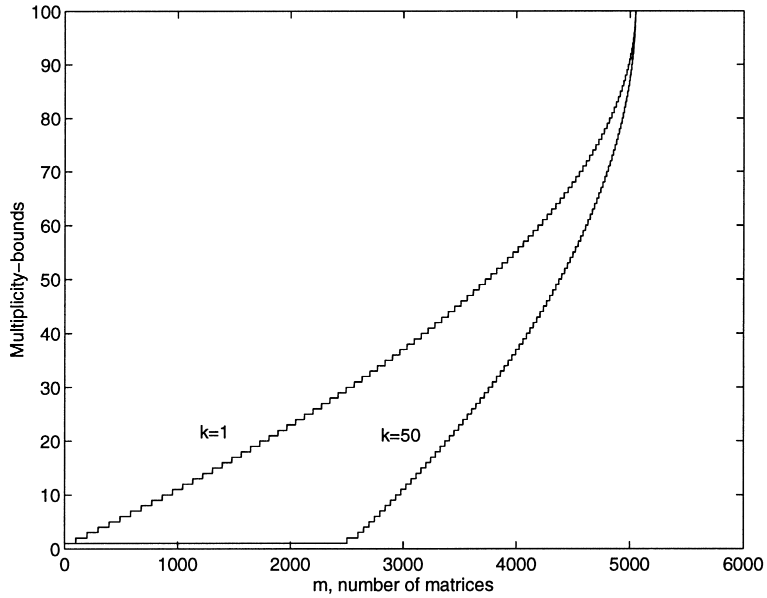

FIGURE 1. Multiplicity-bounds for $n = 100$

REMARK 4.7. It is interesting to note that for $k = 1$ there are very few classes of problems where Theorem 4.3 does *not* ensure a multiple first eigenvalue. Consider

$$
\begin{aligned}
&\operatorname{Min} \lambda_1(A_0 + D) \\
&\text{s.t.} \quad D \text{ diagonal}, \\
&I \bullet D = 0.
\end{aligned}
\tag{4.38}
$$

Problem (4.38) arises in combinatorial optimization: when A_0 is chosen as the *Laplacian matrix* of a graph, it yields a relaxation of the *maximum cut problem*; see, e.g., Delorme and Poljak (1993). In this paper the authors show that the solution of (4.38) is always unique. An equivalent problem can be obtained by writing D as a linear combination of $m = n - 1$ matrices, hence (4.30) does *not* imply a multiple first eigenvalue. Indeed, there are instances of (4.38), where the first eigenvalue of the optimal matrix is simple. On the other hand, most other eigenvalue-optimization problems with $k = 1$ appearing in the literature have $m \geq n$, therefore the existence of a multiple first eigenvalue is guaranteed. For another example where $m < n$ (in fact $m = 1$) see Rendl and Wolkowicz (1997).

REMARK 4.8. *Upper* bounds on $\operatorname{mult}(\lambda_k(A(x^*)))$ are established in Shapiro and Fan (1995). These bounds hold *generically*, that is, the subset of matrices $\{A_0, A_1, \ldots, A_m\}$ for which $\operatorname{mult}(\lambda_k(A(x^*)))$ exceeds the upper bound forms a subset of $(\mathcal{S}^n)^{m+1}$, with Lebesgue measure zero. Results of similar flavor were obtained in Alizadeh, Haeberly, and Overton (1997). They show a generic *lower* bound on the rank of extreme matrices in semidefinite programs.

5. Multiplicity of optimal eigenvalues: The general case. Consider the problem

$$
\min\{ f_k(A(x)) : x \in \mathcal{R}^m \}
\tag{5.39}
$$

where $A : \mathcal{R}^m \to \mathcal{S}^n$ is a smooth function.

LEMMA 5.1. *Let x^* be an optimal solution to* (5.39) *with optimal value f_k^*. Define*

$$A_0 = A(x^*),$$

$$A_i = \partial A(x^*)/\partial x_i \quad (i = 1, \ldots, m).$$

Then A_0 minimizes f_k on the affine subspace

$$\left\{ A_0 + \sum_{i=1}^{m} y_i A_i \,|\, y \in \mathcal{R}^m \right\}.$$

PROOF. The problem (5.39) can be rewritten as

$$\begin{aligned} &\min f_k(X) \\ &\text{s.t.} \quad X \in C, \end{aligned} \tag{5.40}$$

where $C = \{A(x) | x \in \mathcal{R}^m\}$. As A_0 is an optimal solution of (5.40) and f_k is locally Lipschitz at A_0, a necessary condition of nonsmooth optimization (see Clarke 1990, p. 52) implies

$$0 \in \partial f_k(A_0) + N_C(A_0). \tag{5.41}$$

Here $\partial f_k(A_0)$ denotes the generalized gradient of f_k at A_0, which by the convexity of f_k reduces to its subdifferential. The set $N_C(A_0)$ is the normal cone of the (not necessarily convex) set C at A_0 defined as (see Clarke 1990, p. 11).

$$N_C(A_0) = \{Y | Y \bullet V \leq 0 \text{ for all } V \in T_C(A_0)\} \tag{5.42}$$

with $T_C(A_0)$ being the tangent cone of C at A_0 in the Clarke sense (see the same reference). However, as C is a smooth manifold, $T_C(A_0)$ reduces to the usual tangent space of C at A_0 translated to the origin (see, e.g., Aubin and Frankowska 1990, p. 151), that is

$$T_C(A_0) = \left\{ \sum_{i=1}^{m} y_i A_i \,|\, y \in \mathcal{R}^m \right\}.$$

But now condition (5.41) is sufficient to guarantee that A_0 minimizes f_k on $A_0 + T_C(A_0)$. □

As a corollary we obtain

THEOREM 5.2. *Assume $k < n$. Let x^* be an optimal solution to* (5.39) *and define A_0, A_i $(i = 1, \ldots, m)$ as in Lemma* 5.1. *Suppose that f_k is strictly convex at A_0 on the affine subspace*

$$\left\{ A_0 + \sum_{i=1}^{m} y_i A_i \,|\, y \in \mathcal{R}^m \right\}. \tag{5.43}$$

Then

$$\lambda_k(A_0) = \lambda_{k+1}(A_0), \text{ and} \tag{5.44}$$

$$\operatorname{mult}(\lambda_k(A_0)) \geq n - \tau(t(n) - m - 1, k - 1, n - k - 1). \tag{5.45}$$

PROOF. Define the function A_{x^*} as

$$A_{x^*}(y) = A_0 + \sum_{i=1}^{m} y_i A_i \quad (y \in \mathcal{R}^m).$$

Let $\mathcal{O}$ denote the set of optimal solutions of the linearized problem

$$\mathcal{O} = \{y \in \mathcal{R}^m \mid f_k(A_{x^*}(y)) = f_k(A_0)\}. \tag{5.46}$$

By Lemma 5.1, $0 \in \mathcal{O}$. But the strict convexity of f_k on the affine subspace (5.43) at $A_0 = A_{x^*}(0)$ is equivalent to the strict convexity of $f_k \circ A_{x^*}$ at 0. Therefore 0 is an extreme point of $\mathcal{O}$, and Theorem 4.3 implies the clustering of eigenvalues at $A_0 = A_{x^*}(0)$. □

Theorem 5.2 clarifies that the reason that causes multiple eigenvalues to occur is not A being affine; rather it is the strict convexity assumption being satisfied. On the other hand, the existence of an optimal solution x^* that would satisfy this assumption cannot be guaranteed in general, when A is not affine.

6. Minimizing the sum of eigenvalues in absolute value. Let $B \in \mathcal{S}^n$, $k \leq n$. Let B have eigenvalues $\mu_1(B), \ldots, \mu_n(B)$ arranged in such a way that $|\mu_1(B)| \geq \cdots \geq |\mu_n(B)|$. Define

$$g_k(B) = \sum_{i=1}^{k} |\mu_i(B)|. \tag{6.47}$$

Let $A : \mathcal{R}^m \to \mathcal{S}^n$ be a smooth function. In this section we study the problem

$$\min\{g_k(A(x)) : x \in \mathcal{R}^m\}. \tag{6.48}$$

For duality theory, algorithms, and applications we refer to Overton (1992) and Overton and Womersley (1993).

The clustering phenomenon also occurs in optimal solutions of (6.48). Specifically, if x^* is an optimal solution, then frequently $|\mu_k(A(x^*))| = |\mu_{k+1}(A(x^*))|$, and the eigenvalues attaining $|\mu_k(A(x^*))|$ can appear more than twice in the spectrum of $A(x^*)$ (on either side, or both sides).

In this section we outline how the ideas presented in the previous sections can be used to explain the clustering of eigenvalues in (6.48).

First, we consider two simple SDP's to determine $g_k(B)$ for a fixed $B \in \mathcal{S}^n$

$$\begin{aligned} &\text{Min } kz + I \bullet (V + W) \\ &\text{s.t.} \quad V, W, S, T \succeq 0, \\ &zI + V - W = B, \\ &zI + S - T = -B, \end{aligned} \tag{6.49}$$

and

$$\begin{aligned} &\text{Max } B \bullet (X - Y) \\ &\text{s.t.} \quad I \succeq X,\ Y \succeq 0, \\ &I \bullet (X + Y) = k. \end{aligned} \tag{6.50}$$

As shown in Alizadeh (1995) (6.49) and (6.50) are dual SDPs with equal optimal value. In Overton and Womersley (1993) the optimal solutions of (6.50) are determined. Here we give the analogous result for (6.49). The proof of the following theorem is similar to the proof of Theorem 3.3, hence omitted.

THEOREM 6.1. *Consider the SDP* (6.49). *Write* $B = Q\Lambda Q^T$ *with* Q *an* n *by* n *orthonormal matrix*, $\Lambda = \text{Diag } \lambda$. *Let* $\{\mu_1, \ldots, \mu_n\}$ *be a permutation of* $\{\lambda_1, \ldots, \lambda_n\}$ *s.t.* $|\mu_1| \geq \cdots \geq |\mu_n|$, *and assume*

- $k_1, k_2 \geq 0$, $k_1 + k_2 = k$.
- $\lambda_1 \geq \cdots \geq \lambda_{k_1} \geq |\mu_k|$ *and* $-\lambda_n \geq \cdots \geq -\lambda_{n-k_2+1} \geq |\mu_k|$.

Then the optimal value of (6.49) *is* $\sum_{i=1}^k |\mu_i|$ *and* $(z^*, V^*, W^*, S^*, T^*)$ *is optimal if and only if*

$$|\mu_{k+1}| \leq z^* \leq |\mu_k|, \tag{6.51}$$

$$V^* = Q(\text{Diag } v^*)Q^T, \qquad W^* = Q(\text{Diag } w^*)Q^T, \tag{6.52}$$

$$S^* = Q(\text{Diag } s^*)Q^T, \qquad T^* = Q(\text{Diag } t^*)Q^T, \tag{6.53}$$

where

$$\begin{aligned} v^* &= (\lambda_1 - z^*, \ \ldots, \ \lambda_{k_1} - z^*, \ 0, \ \ldots, \ 0)^T \\ w^* &= (0, \ \ldots, \ 0, \ z^* - \lambda_{k_1+1}, \ \ldots, \ z^* - \lambda_n)^T \\ s^* &= (-\lambda_n - z^*, \ldots, -\lambda_{n-k_2+1} - z^*, \ 0, \ \ldots, \ 0)^T \\ t^* &= (0, \ \ldots, \ 0, \ z^* - (-\lambda_{n-k_2}), \ldots, z^* - (-\lambda_1))^T. \quad \square \end{aligned} \tag{6.54}$$

Analogously to the remark following Theorem 3.3, it can be shown that the set of optimal solutions of (6.49) is uniquely (up to the choice of z) determined by B; we shall denote this set by $\Theta_k(B)$.

Consider the case when the function A is affine. Assume $m \geq 1$ and for $x \in \mathcal{R}^m$ define

$$A(x) = A_0 + \sum_{i=1}^m x_i A_i$$

where $A_0, A_1, \ldots, A_m \in \mathcal{S}^n$ and we assume w.l.o.g. that $A_1, \ldots, A_m$ are linearly independent. Substituting $B = A(x)$ in (6.49) yields the SDP-formulation for the affine problem

$$
\begin{aligned}
&\operatorname*{Min}_{x,z,V,W,S,T} \quad kz + I \bullet (V + W) \\
&\text{s.t.} \quad V, W, S, T \succeq 0, \\
&zI + V - W = A(x), \\
&zI + S - T = -A(x)
\end{aligned}
\tag{6.55}
$$

(derived in Alizadeh 1995).

The set of optimal solutions of (6.48) is closed and convex. Moreover, it is also bounded (since g_k has bounded level sets) hence it has at least one extreme point. Let us introduce the notation

$$\underline{k} = \lfloor \tfrac{k}{2} \rfloor, \quad \overline{k} = \lceil \tfrac{k}{2} \rceil.$$

The analogue of Theorem 4.3 is

THEOREM 6.2. *Let x^* be an extreme point of the set of optimal solutions of* (6.48), *and assume*

$$m > \underline{k}(n - \overline{k}) + \overline{k}(n - \overline{k}).$$

Then

$$|\mu_k(A(x^*))| = |\mu_{k+1}(A(x^*))|. \tag{6.56}$$

PROOF OUTLINE. Let

$$F^* = \{x^*\} \times \Theta_k(A(x^*)).$$

Similarly to the proof of 4.2 one can show that F^* is a face of the feasible set of (6.55) of dimension 1, when $|\mu_k(A(x^*))| > |\mu_{k+1}(A(x^*))|$ and 0 otherwise.

To obtain a contradiction, suppose $|\mu_k(A(x^*))| > |\mu_{k+1}(A(x^*))|$. Let $(x^*, z, V, W, S, T) \in F^*$ with $z = \frac{1}{2}(|\mu_k(A(x^*))| + |\mu_{k+1}(A(x^*))|)$. Define k_1 and k_2 as in Lemma 6.1. Then we must have

$$\operatorname{rank} V = k_1, \quad \operatorname{rank} W = n - k_1, \quad \operatorname{rank} S = k_2, \quad \operatorname{rank} T = n - k_2. \tag{6.57}$$

On the other hand, in the feasible set of (6.55) the number of unconstrained variables is $m + 1$, and the number of equality constraints is $2t(n)$. Therefore, Theorem 2.2 implies

$$t(k_1) + t(n - k_1) + t(k_2) + t(n - k_2) \le 2t(n) - (m + 1) + 1 \Leftrightarrow$$

$$m \le k_1(n - k_1) + k_2(n - k_2). \tag{6.58}$$

The right-hand side of inequality (6.58) is maximized when $k_1 = \underline{k}$, $k_2 = \overline{k}$. Hence if m is greater than $\underline{k}(n - \underline{k}) + \overline{k}(n - \overline{k})$ then (6.57) is impossible; thus $|\mu_k(A(x^*))| = |\mu_{k+1}(A(x^*))|$ follows, as needed. □

It is shown in Overton and Womersley (1993) that the function g_k is nondifferentiable at the matrix B, if and only if $|\mu_k(B)| > |\mu_{k+1}(B)|$. Hence Theorem 6.2 implies that g_k is nonsmooth at $A(x^*)$.

The subdifferential of the *composite function* $g_k \circ A$ at x^* can be computed as (see the above references)

$$\partial(g_k \circ A)(x^*) = \{(A_1 \bullet U, \ldots, A_m \bullet U)^T \,|\, U \in \partial g_k(A(x^*))\}.$$

Therefore $g_k \circ A$ will *generally* be nonsmooth at x^* when g_k is nonsmooth at $A(x^*)$. An example showing that this is not always the case is when $A(x) \equiv I$.

It is interesting to note that the threshold value of m in Theorem 6.2 that is needed to ensure nondifferentiability of g_k at $A(x^*)$ is roughly the half of the threshold value required for f_k (cf. Theorem 4.3). Also, it is possible to derive a lower bound on the ''multiplicity'' of $|\mu_k(A(x^*))|$; precisely, on the number of appearances of the eigenvalues attaining $|\mu_k(A(x^*))|$ in the spectrum of $A(x^*)$. The lower bound is an increasing function of m.

Furthermore, when A is a not necessarily affine, smooth function, Lemma 5.1 is true when f_k is replaced by g_k; hence a result analogous to Theorem 5.2 can be proven for the function g_k.

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References

Alizadeh, F. (1995). Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM J. Optim.* **5** 13–51.

———, J.-P. Haeberly, M. Overton (1997). Complementarity and nondegeneracy in semidefinite programming. *Math. Programming* **77** 111–128.

Aubin, J.-P., H. Frankowska (1990). *Set-Valued Analysis*, Birkhäuser, Boston.

Barker, G. P., D. Carlson (1975). Cones of diagonally dominant matrices, *Pacific J. Math*., **57** 15–32.

Clarke, F. H. (1990). *Optimization and Nonsmooth Analysis*, Wiley, New York.

Cullum, J., W. E. Donath, P. Wolfe (1975). The minimization of certain nondifferentiable sums of eigenvalue problems. *Math. Programming Stud.* **3** 35–55.

Delorme, C., S. Poljak (1993). Laplacian eigenvalues and the maximum cut problem. *Math. Programming* **62** 557–574.

Hiriart-Urruty, J.-B., D. Ye (1995). Sensitivity analysis of all eigenvalues of a symmetric matrix. *Numer. Math.* **70** 45–72.

Horn, R. A., C. R. Johnson (1987). *Matrix Analysis*, Cambridge University Press.

Karger, D. R. Motwani, M. Sudan (1994). Approximate graph coloring by semidefinite programming. *Proc. of* 35*th IEEE Symp. on Foundations of Computer Science*, 2–13.

Ky Fan (1949). On a theorem of Weyl concerning the eigenvalues of linear transformations. *Proc. Nat. Acad. Sci. USA* **35** 652–655.

Lewis, A. S. (1996). Convex analysis on the Hermitian matrices. *SIAM J. Optim.* **6** 164–177.

———, M. Overton (1996). Eigenvalue optimization. *Acta Numerica* **5** 149–190.

Nesterov, Yu., A. Nemirovskii (1994). *Interior Point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia.

Overton, M. L. (1992). Large-scale optimization of eigenvalues. *SIAM J. Optim.* **2** 88–120.

———, R. S. Womersley (1992). On the sum of the largest eigenvalues of a symmetric matrix. *SIAM J. Matrix Anal. Appl.* **13** 41–45.

———, ——— (1993). Optimality conditions and duality theory for minimizing sums of the largest eigenvalues of symmetric matrices. *Math. Programming* **62** 321–357.

Pataki, G. (1994). *On the facial structure of cone-LPs and semidefinite programs*. Management Science Research Report MSRR-#595, Graduate School of Industrial Administration, Carnegie Mellon University, Pittsburgh, PA.

Ramana, M. V., A. Goldman (1995). Some geometric results in semidefinite programming. *J. Global Opt.* **7** 33–50.

Rendl, F., H. Wolkowicz (1997). A semidefinite framework for trust region subproblems with applications to large scale minimization. *Math. Programming* **77** 273–299.

Rockafellar, R. T. (1970) *Convex Analysis*, Princeton University Press, Princeton, New Jersey.

Shapiro, A. (1985). Extremal problems on the set of nonnegative definite matrices. *Linear Algebra Appl.* **67** 7–18.

———, M. K. H. Fan (1995). On eigenvalue optimization. *SIAM J. Optim*. **5** 552–569.
Stoer, J., C. Witzgall (1970). *Convexity and Optimization in Finite Dimensions*, Springer-Verlag, New York.
Vandenberghe, L., S. Boyd (1996). Semidefinite programming. *SIAM Rev*. **38** 49–95.
Wolkowicz, H. (1981). Some applications of optimization in matrix theory. *Linear Algebra Appl*. **40** 101–118.

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