Slide 1

Column Basis Reduction, Decomposable Knapsack and Cascade Problems

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Slide 2

What is basis reduction ?

Given integral matrix A, basis reduction (BR) computes a unimodular $U (\Leftrightarrow \det U = \pm 1)$ st. the columns of AU are "short" and "nearly" orthogonal.

Example

$$A = \begin{pmatrix} 289 & 18 \\ 466 & 29 \\ 273 & 17 \end{pmatrix}, \; U = \begin{pmatrix} 1 & -15 \\ -16 & 241 \end{pmatrix}, \; AU = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

Computing AU $\Leftrightarrow$ doing *elementary column operations* on A:

- adding an integer multiple of a column to another; multiplying a column by -1; swapping columns.

Slide 3

Reformulating equality constrained IP feasibility problems

Aardal, Hurkens, Lenstra (1998); Aardal, Bixby, Hurkens, Lenstra, Smeltink (1999); Aardal, Lenstra (2004); Louvaux, Wolsey (2003).

$$\begin{array}{c} x \in \mathcal{Z}^n \\ Ax = d \\ \ell \le x \le u \\ \downarrow \qquad \textbf{Reformulation} \\ \lambda \in \mathcal{Z}^{n-m} \\ \ell \le B\lambda + x_d \le u \end{array}$$

Slide 4

Here

$$\{\, x \in \mathcal{Z}^n \mid Ax = d \,\} \quad = \quad \{\, x_d + B\lambda \mid \lambda \in \mathcal{Z}^{n-m} \,\}$$

- $[B, x_d]$ is
  - integral, columns are short and nearly orthogonal.
  - found by doing **basis reduction** on an enlarged matrix using two large constants N_1, N_2.
- The reformulated problem of finding

$$\lambda \in \mathcal{Z}^{n-m},\ \ell \le B\lambda + x_d \le b$$

  proved experimentally *much* easier to solve for some problems, e.g. the Cornuejols-Dawande instances.

Slide 5

Questions

1. Why only equality constrained problems?
2. Why does it work?

Slide 6

Rest of talk

1. Column BR: simplified reformulation for arbitrary IPs. 2 variants: in range space and null space.
2. Computational study.
3. Analysis for a general problem class, called *decomposable knapsack problems*.

Slide 7

Rangespace reformulation

$$\begin{aligned} P &= \{\, x \mid \ell \le \ Ax \le b \,\} \\ \tilde{P} &= \{\, y \mid \ell \le (AU)y \le b\} \end{aligned}$$

where U is unimodular.

There is 1-1 correspondence between

$$P \cap \mathcal{Z}^n \text{ and } \tilde{P} \cap \mathcal{Z}^n$$

given by

$$Uy = x$$

We choose U so columns of AU are reduced. We can do the same if some of the "$\le$" are actually "$=$".

Slide 8

Nullspace reformulation

If

$$A_1 x = b_1$$

is a subset of the inequalities in $\ell \le Ax \le b$, then

$$\{\, x \in \mathcal{Z}^n \mid A_1 x = b_1 \,\} = \{\, x_d + B_1 \lambda \mid \lambda \in \mathcal{Z}^{n-m} \,\}$$

$[B_1, x_d]$ is found by a Hermite Normal Form (HNF) computation; columns are *not* in general short and orthogonal.

Substitute $B_1\lambda + x_d$ for x, and do the rangespace reformulation.

If all constraints are equalities, then essentially equivalent to the Aardal et al. reformulation.

Slide 9

- Such a simple reformulation actually works for essentially all hard IPs used to test "nontraditional" IP algorithms!
- We need a problem class on which we can *analyze* its action.

Slide 10

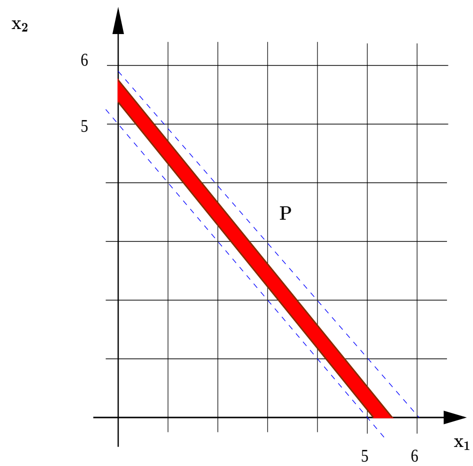
Branching on a constraint

Given polyhedron P, integral vector c,

- $\mathrm{width}(c, P) = \max\{\, cx \mid x \in P \,\} - \min\{\, cx \mid x \in P \,\}$.
- **branching on** cx means creating the branches $cx = \lceil \min \rceil$, $cx = \lceil \min \rceil + 1, \ldots, cx = \lfloor \max \rfloor$.
- If the interval $[\min, \max]$ contains no integer, then P contains no integral point.

Slide 11

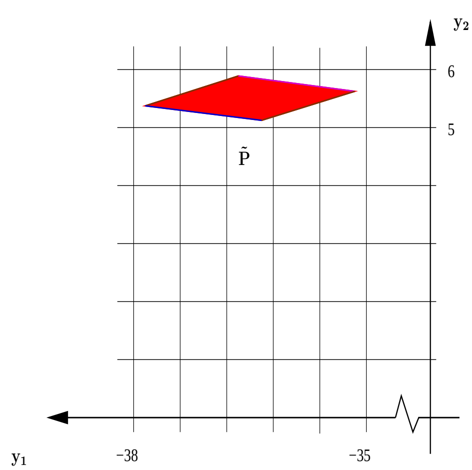
Example :
$$\begin{aligned} 106 \leq \quad 21x_1 + 19x_2 \quad &\leq 113 \\ x_1, x_2 \in \quad &\in [0,6] \cap \mathcal{Z} \end{aligned}$$



Hard for branching on x_is.

Easy for branching on $x_1 + x_2$: max = 5.94, min = 5.04.

Slide 12



After reformulation: branching on y_2 proves infeasibility.

Slide 13

2-level decomposable knapsack problems

The example is an instance of

$$(KP_2) \quad \beta' \leq a\,x\; \leq \beta, \qquad 0 \leq x \leq u, \quad x \in \mathcal{Z}^n,$$

where

- $a = pM + r$, with $p \in \mathcal{Z}^n_+$, $r \in \mathcal{Z}^n$; M large;
- β, β' chosen, so KP_2 is LP-feasible, IP-infeasibility proven by branching on px.
- In the example, $(21, 19) = (1, 1) * 20 + (1, -1)$.

Slide 14

What does the reformulation do on these?

Recall general reformulation:

$$P = \{x \,|\, \ell \leq Ax \leq b\,\} \Leftrightarrow \tilde{P} = \{y \,|\, \ell \leq\; (AU)y \leq b\}$$

Slide 15

Basis reduction in range space

We choose U unimodular, s.t.

$$\begin{pmatrix} pM + r \\ I \end{pmatrix} U \quad \text{is reduced.}$$

Theorem: M suff. large $\Rightarrow$

$$pU = (\,\overbrace{0\ \ldots\ 0}^{n-1}\ \alpha\,) \quad \text{for some } \alpha \in \mathcal{Z} \setminus \{\,0\,\}.$$

Corollary:

$$U\,y = x \;\Rightarrow\; pUy \;=\; p\,x \;\;\Rightarrow\; \alpha y_n = \; p\,x$$

$\Rightarrow$ branching on y_n proves infeasibility.

Slide 16

"Sufficiently large" means:

- If LLL (Lenstra, Lenstra, Lovasz) reduction is used, $M > 2^{n+1}\,\|\,p\,\|\|\,r\,\|^2$.
- If KZ (Korkhine-Zolotarev) reduction is used, $M > \sqrt{n}\,\|\,p\,\|\|\,r\,\|^2$.

Slide 17

Basis reduction in null space

Can be used if $\beta = \beta'$ $\rightarrow$ reformulation has $n-1$ variables.

We can similarly prove: M suff. large $\Rightarrow$ branching on y_{n-1} in reformulation $\equiv$branching on px in original problem.

Slide 18

A classic example of a decomposable knapsack problem: Jeroslow's problem

$$\begin{aligned} 2(x_1 + \ldots + x_n) &= n \\ x_i \in \{\, 0, 1 \,\}^n & \end{aligned}$$

where n is odd. In B&B branching on the x_i no node is pruned above level $n/2$. If we branch on $x_1 + \ldots x_n$, we solve it at the root.

Here $p = e$, $r = 0$, $M = 2$.

Slide 19

Other examples:

1. $p = e$, $r = (2^0, \ldots, 2^{n-1})$, $u = e$, $M = 2^{n+\ell+1}$: Todd's problem from Chvátal "Hard knapsack problems" (1983).
2. $p = e$, $r = (1, \ldots, n)$, $u = e$, $M = n(n+1)$: Avis' problem from same paper.
3. A modification of (1): Gu, Nemhauser (2001).
4. $p \geq 0$, r arbitrary, $u = +\infty$, $\beta = \beta'$: Aardal-Lenstra Frobenius problems.

Out of these: (1) and (2) take $2^{n/2}$ nodes for ordinary B&B ; in (4) has a $\beta = \text{const}^* M^2$ for which problem is infeasible.

Slide 20

Algorithms that find thin directions to branch on

- H. W. Lenstra (1983); Kannan (1987); Eisenbrand (2004): polytime algorithms for IP in fixed dimensions. Implementation: Gao, Zhang (2002); Modification and implementation: Mehrotra, Li (2004).
- Generalized BR: Lovasz, Scarf (1990); Implementation: Cook, Rutherford, Scarf, Shallcross (1993); Modification and implementation: Mehrotra, Li (2004).

Slide 21

When thinner $\neq$ better

$$5660 \leq 520x_1 + 725x_2 + 1156x_3 + 1574x_4 + 1794x_5 + 1829x_6$$
$$+2023x_7 + 2221x_8 + 2267x_9 + 2465x_{10} + 2496x_{11} \leq 5661$$
$$x_i \in \{0,1\} \ (i = 1, \ldots, 11). \tag{1}$$

- IP-infeasible, and 'reasonably" hard for B&B .
- If Q = LP relaxation, then $\min_{c \text{ integral}} \text{width}(c, Q) = 1 - 0$, attained at e_i.
- $\exists\, p_1$ integral: $\text{width}(p_1, Q) = 25.34 - 24.30 \Rightarrow$ constraint $p_1 x = 25$ can be added to LP.
- If Q' = new LP relaxation, then $\exists\, p_2$ integral: $\text{width}(p_2, Q') = 14.93 - 14.02 \Rightarrow$ proves IP-infeasibility.

Slide 22

- So, a direction with width $= 1.04$ beats all directions with width 1!
- Such problems are called *cascade* problems: branching on a good direction has a "cascade" effect.
- There are more extreme examples, with width in good direction ≈ 1.5.

Slide 23

$t+1$-level decomposable knapsack problems

- For $a = p_1 M_1 + p_2 M_2 + \ldots + p_t M_t + r$, with $M_1 > M_2 > \ldots > M_t$ and suitable β, β'

$$(KP_{t+1}) \quad \beta' \leq a\,x \leq \beta, \quad 0 \leq x \leq u, \quad x \in \mathcal{Z}^n$$

Problem is

- easy, if branching on $p_1 x$, $p_2 x$, ..., $p_t x$.
- hard, if branching on x_j variables, if parameters suitably chosen.
- cascade problems can be constructed this way.

Slide 24

When using the rangespace reformulation: compute U so that

$$\begin{pmatrix} \sum_{i=1}^t p_i M_i + r \\ I \end{pmatrix} U \quad \text{is reduced.}$$

Theorem: If separation between $M_1 > M_2 > \ldots > M_t$ is suitably large, then

$$\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_t \end{pmatrix} U = \begin{pmatrix} 0 & 0\ldots & 0 & 0 & 0 & * \\ 0 & 0\ldots & 0 & 0 & * & * \\ \vdots & & & & & \\ 0 & 0\ldots & * & \ldots & * & * \end{pmatrix}$$

Remark: When computing U, we do not know the decomposition!!

Slide 25

Corollary: Branching on y_n, $y_{n-1}, \ldots, y_{n-t}$ in reformulation

$\Leftrightarrow$ branching on $p_1 x$, $p_2 x$, ..., $p_t x$ in original problem.

Analogous result for nullspace reformulation.

- That is, column BR
  - takes the *unknown* "dominant" branching combinations;
  - transforms them into individual variables;
  - lines them up in reverse order of significance!

Slide 26

Computational results

- BR: by NTL library of Victor Shoup.
- IP solver: CPLEX 9.0.
- Machine: 3.2 GHz Linux PC.
- We adapted column BR to deal with optimization problems.
- We report: time and B&B nodes taken by CPLEX 9.0 *after* reformulation.
- We do not report: time taken *without* reformulation (even in the simplest case, it is a few hundred thousand B&B nodes; usually it is $+\infty$).

Slide 27

To solve

$$\begin{array}{rrcl} \max & cx & & \\ st. & Ax & \leq & b \\ & x & \in & \mathcal{Z}^n \end{array}$$

we replace A with AU, c with cU, where U makes

$$\begin{pmatrix} c \\ A \end{pmatrix} U$$

reduced.

Slide 28

Maximization versions of integer subset sum

$$\begin{array}{rrcl} \max & ax & & \\ st. & ax & \leq & \beta \\ & x & \in & \mathcal{Z}^n_+. \end{array} \tag{2}$$

First four instances from Cornuéjols, Urbaniak, Weismantel, Wolsey (1998). Last (shown below) from Wolsey: Integer Programming (1999).

$$(12228, 36679, 36682, 48908, 61139, 73365); 89716837$$

Number of B&B nodes after column BR: 5, 0, 9, 0, 10.

Slide 29

Feasibility versions of same instances

For (a, β), $\beta_a :=$ optimal value. Then check the feasibility of

$$\begin{aligned} ax &= \beta_a \\ x &\in \mathcal{Z}^n_+, \end{aligned} \tag{3}$$

using 1) rangespace reformulation, 2) nullspace reformulation. Number of B&B nodes is between 0 and 10 for all 5 instances, for both choices.

Same happens, if rhs is chosen as $\beta_a + \gcd(a)$.

Slide 30

Marketshare problems (Cornuéjols, Dawande)

We need to find

$$x \in \{0, 1\}^n, \quad Ax = d,$$

where $m = 6$ or $m = 7$, $n = 10(m-1)$. A, d are generated to make the problem difficult.

	range space		null space	
	# BB	CPU	# BB	CPU
ms1	288597	175.30	51887	32.80
ms2	220803	165.40	52920	43.70

Slide 31

Relaxed marketshare problems

Same data, but we want to find

$$x \in \{0, 1\}^n, \quad d - 1 \leq Ax \leq d.$$

After column BR

- *markshare1:* $85,466$ nodes, 53 seconds; *markshare2:* $250,368$ nodes, 211 seconds.

Slide 32

$\boxed{\textbf{Cascade2}}$

The "big brother" of the 11-variable instance.

- $n = 100$ variables, $a_j \leq 14,000$, β, $\beta' \leq 100,000$.
- Original problem does not solve by CPLEX after enumerating 2 billion B&B nodes.
- Easy, if we branch on $p_1 x$, then $p_2 x$.
- Reformulation solves at rootnode.

Slide 33

Caveats

- There are hard IPs for which the reformulation does *not* work :-(
- The reformulation uncovers the hidden “dominant” directions in the polyhedron - but in some hard problems, these may not exist, if the problem is symmetric.

Slide 34

Conclusions and further work

- A general, and very simple reformulation technique for arbitrary IPs.
- A fairly general class of IPs that are provably hard for ordinary B&B .
- Analysis: the provably hard problems turn into provably easy ones: the reformulation “uncovers” the hidden, dominant directions.
- The *cascade* problems: thinner $\neq$ better!
- Works well in on most small, hard IPs from the literature.