A Simple Derivation of a Facial Reduction Algorithm and Extended Dual Systems

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Abstract

The Facial Reduction Algorithm (FRA) of Borwein and Wolkowicz and the Extended Dual System (EDS) of Ramana aim to better understand duality, when a conic linear system

$$Ax \leq_K b \qquad (P)$$

has no strictly feasible solution. We

- provide a simple proof of the correctness of a variant of FRA.
- show how it naturally leads to the validity of a family of extended dual systems.
- Summarize, which subsets of K related to the system (P) (as the minimal cone and its dual) have an extended representation.

1 Introduction

Farkas' lemma assuming a CQ Duality results for the conic linear system

$$Ax \leq_K b \qquad (P)$$

are usually derived assuming some *constraint qualification (CQ)*. The most frequently used CQ is *strict feasibility*, ie. assuming the existence of a $\bar{x}$ with $A\bar{x} <_K b$. Here K is a closed convex cone, $A : X \rightarrow Y$ a linear operator, with X and Y being euclidean spaces. We write $z \leq_K y$, and $z <_K y$ to mean that $y - z$ is in K, or in $\operatorname{ri} K$, respectively.

Let K^* be the dual cone of K. A fundamental result that implies both strong duality between a primal-dual pair of conic linear programs, and the existence of a certificate of infeasibility of a conic linear system (again, assuming an appropriate CQ) is

Theorem 1.1. (Farkas' lemma) *Suppose that (P) is strictly feasible, $c \in X$, $c_0 \in \mathbb{R}$. Then for all x feasible solutions of (P) $\langle c, x\rangle \leq c_0$ holds, iff there is a y such that*

$$y \geq_{K^*} 0,\ A^*y = c,\ \langle b, y\rangle \leq c_0. \tag{1.1}$$

■

Two approaches are known to derive strong duality results for conic linear systems without assuming a CQ, and to better understand systems which are not strictly feasible.

The Facial Reduction Algorithm Borwein and Wolkowicz in [2, 3] note that (P) is always equivalent to a strictly feasible system

$$Ax \leq_{F_{\min}} b,$$

where $F_{\min}$ is a face of K, called the *minimal cone* of (P). Therefore, as $F_{\min}$ is a closed convex cone, Theorem 1.1 holds without requiring a CQ, if we replace $y \geq_{K^*} 0$ with

$$y \geq_{F_{\min}^*} 0.$$

The technique of deriving duality results using the minimal cone is called *facial reduction*. Furthermore, they provide an algorithm to construct a sequence of faces $K = F_0 \supseteq \cdots \supseteq F_t = F_{\min}$ for some $t \geq 0$. We shall call their method a Facial Reduction Algorithm (FRA).

An Extended Dual System For a semidefinite linear system, i.e. when $K = K^* = \mathcal{S}^n_+$, Ramana in [15] has developed the approach of an *Extended Dual System (EDS)*. (The term used by him was an "Extended Lagrange-Slater Dual"; we feel that our terminology is better suited for the treatment presented in this paper.) Essentially, he has shown that there is a set $\operatorname{ext}(A, b, K^*)$, such that Theorem 1.1 holds without any CQ assumption if we replace $y \geq_{K^*} 0$ with

$$(y, w) \in \operatorname{ext}(A, b, K^*) \text{ for some } w. \tag{1.2}$$

Moreover, $\text{ext}(A, b, K^*)$ is the set of feasible solutions of a conic linear system in which the only "nontrivial" (ie. different from a direct product of copies of $\mathbb{R}$, and $\mathbb{R}_+$) cone is $K = K^*$.

Related Literature The resemblance between these results is not coincidental: Ramana, Tuncel and Wolkowicz have shown in [14] that FRA and EDS are closely related. Alternative interpretations of extended dual systems were given by Luo, Sturm and Zhang in [10], and Kortanek and Zhang in [7]. An interesting, and novel application of FRA was introduced by Sturm in [17]: deriving error bounds for semidefinite systems that have no strictly feasible solution. Luo and Sturm generalized this approach to mixed semidefinite, and second order conic systems; see [9]. Facial reduction was used by several other authors to derive duality results without a CQ assumption; see Lewis [8].

A Unified Treatment Our aim is to provide a unified and transparent derivation of FRA and EDS. Precisely, under the assumption that

$$F^* = K^* + F^\perp \text{ for all faces } F \text{ of } K, \tag{1.3}$$

we

(1) give a proof of the correctness of a variant of FRA.

(2) show that it immediately implies that

$$F^*_{\min} = \{\, y \mid (y, w^1) \in \text{ext}_1(A, b, K^*, T) \text{ for some } w^1 \,\}.$$

Here $\text{ext}_1(A, b, K^*, T)$ is the feasible set of a conic linear system that depends on A, b, K^* and T, a closed convex cone which is related to K^*. In other words, $F^*_{\min}$ has an *extended formulation.*

(3) Prove that when $K = K^* = \mathcal{S}^n_+$, the dependence on T can be eliminated; a similarly described $\text{ext}_2(A, b, K^*)$ can be found, so that

$$F^*_{\min} = \{\, y \mid (y, w^2) \in \text{ext}_2(A, b, K^*) \text{ for some } w^2 \,\}.$$

(4) Survey other results on the representability (in the above sense) of $F_{\min}$ and related sets: its dual cone, orthogonal complement, and complementary face.

We note that assumption (1.3) is satisfied for most cones of interest; see Section 2. To keep the presentation simple, the *only* results that we will use is Theorem 1.1, and some elementary facts from convex analysis.

Section 2 contains all necessary preliminaries. In Section 3 we derive a simple variant of FRA; in Section 4 "translate" the algorithm into an EDS, and show that for the semidefinite case, there is such a system expressed purely in terms of K^*, thus recovering Ramana's result. Section 5 presents variants of extended dual systems, and Section 6 studies the representability of $F_{\min}$ and its related sets.

2 Preliminaries

Operators and matrices Linear operators are denoted by capital letters. If A is a linear operator, then A^* will stand for its adjoint. When a matrix is considered to be an element of a euclidean space, not a linear operator, it is denoted by a small letter.

Convex Sets The open line-segment between points y and z is denoted by (y, z). Let C be a closed convex set. A convex subset F of C is called a *face* of C, and this fact is denoted denoted by $F \trianglelefteq C$, if $x \in F,\ y, z \in C,\ x \in (y, z)$ implies $y, z \in F$. For $x \in C$ we denote by $\operatorname{face}(x, C)$ the minimal face of C that contains x, that is with the property $x \in \operatorname{ri} \operatorname{face}(x, C)$.

For $x \in C$, the set of feasible directions, and the tangent space at x in C are defined as

$$\begin{aligned} \operatorname{dir}(x, C) &= \{\, y \mid x + ty \in C \text{ for some } t > 0 \,\}, \\ \tan(x, C) &= \operatorname{cl} \operatorname{dir}(x, C) \cap - \operatorname{cl} \operatorname{dir}(x, C) \\ &= \{\, y \mid \operatorname{dist}(x \pm ty, C) = o(t) \,\}. \end{aligned}$$

The equivalence of the alternative expressions for $\tan(x, C)$ follows e.g. from [6, page 135].

Cones A convex set K is a *cone*, if $\mu K \subseteq K$ holds for all $\mu \geq 0$. The *dual* of the cone K is

$$K^* = \{\, z \mid \langle z, x \rangle \geq 0 \text{ for all } x \in K \,\}.$$

If $F \trianglelefteq K$, and $\bar{x} \in \operatorname{ri} F$ is fixed, then the *complementary* (or *conjugate*) face of F is defined alternatively as (the equivalence is straightforward)

$$\begin{aligned} F^{\triangle} &= \{\, z \in K^* \mid \langle z, x \rangle = 0 \ \text{ for all } x \in F \,\} \\ &= \{\, z \in K^* \mid \langle z, \bar{x} \rangle = 0 \,\} \end{aligned}$$

The complementary face of $G \trianglelefteq K^*$ is defined analogously, and denoted by $G^{\triangle}$. K is facially exposed, i.e. all faces of K arise as the intersection of K with a supporting hyperplane, iff for all $F \trianglelefteq K$, $F^{\triangle\triangle} = F$, see ([4], Theorem 6.7). For brevity, we write $F^{\triangle *}$ for $(F^{\triangle})^*$, and $F^{\triangle\perp}$ for $(F^{\triangle})^{\perp}$.

A closed convex cone K is called *nice*, if

$$F^* = K^* + F^\perp \text{ for all } F \trianglelefteq K$$

For the purposes of this work, it is enough to note that

- For the FRA to be applicable, the underlying cone must be nice.
- Nice cones are also easier to deal with in other areas of the "duality without CQ" subject, see [13].
- Polyhedral, semidefinite, and second order cones are nice, see [12].
- Nice cones must be facially exposed, see [13].

If K is a cone, $x \in K$, then $\tan(x, K)$ can be conveniently expressed (see [12]) as

$$\tan(x, K) = \operatorname{face}(x, K)^{\triangle\perp} \tag{2.4}$$

We remark, that (2.4) holds for all closed convex cones, not only for nice ones ([11], [13]).

Example 2.1. (The nonnegative orthant) $K = \mathbb{R}^n_+$ is self-dual with respect to the usual inner product of $\mathbb{R}^n$. If $\bar{x} \in K = \mathbb{R}^n_+$, then

$$\begin{aligned} \operatorname{face}(\bar{x}, \mathbb{R}^n_+) &= \{\, x \in \mathbb{R}^n_+ \,|\, x_i = 0 \;\; \forall\, i \; s.t. \; \bar{x}_i = 0 \,\}, \\ \operatorname{face}(\bar{x}, \mathbb{R}^n_+)^\triangle &= \{\, x \in \mathbb{R}^n_+ \,|\, x_i = 0 \;\; \forall\, i \; s.t. \; \bar{x}_i > 0 \,\}. \end{aligned} \tag{2.5}$$

Then (2.4) and (2.5) yield

$$\tan(\bar{x}, \mathbb{R}^n_+) = \{\, y \in \mathbb{R}^n \,|\, y_i = 0 \;\; \forall\, i \; s.t. \; \bar{x}_i = 0 \,\}. \tag{2.6}$$

Example 2.2. (The semidefinite cone) The space of n by n symmetric, and the cone of n by n symmetric, positive semidefinite matrices are denoted by $\mathcal{S}^n$, and $\mathcal{S}^n_+$, respectively. The space $\mathcal{S}^n$ is equipped with the inner product

$$\langle x, z \rangle \;\; := \;\; \sum_{i,j=1}^n x_{ij} z_{ij},$$

and $\mathcal{S}^n_+$ is self-dual with respect to it.

If $\bar{x} \in K = K^* = \mathcal{S}^n_+$, then

$$\begin{aligned} \operatorname{face}(\bar{x}, \mathcal{S}^n_+) &= \; \{\, x \in \mathcal{S}^n_+ \,|\, \mathcal{R}(x) \subseteq \mathcal{R}(\bar{x}) \,\}, \\ \operatorname{face}(\bar{x}, \mathcal{S}^n_+)^\triangle &= \; \{\, x \in \mathcal{S}^n_+ \,|\, \mathcal{R}(x) \subseteq \mathcal{R}(\bar{x})^\perp \,\}, \end{aligned} \tag{2.7}$$

(Barker and Carlson, [1]; for a simple proof, see [12]).

The expressions in (2.7) can be simplified, if we note that $q^T(\text{face}(\bar{x}, \mathcal{S}^n_+))q = \text{face}(q^T\bar{x}q, \mathcal{S}^n_+)$ for any full rank matrix q, therefore, we can assume $\bar{x} = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$. If $\operatorname{rank} \bar{x} = r$, then (2.4) and (2.7) lead to

$$\tan(\bar{x}, \mathcal{S}^n_+) = \left\{ \begin{pmatrix} a & b \\ b^T & 0 \end{pmatrix} \mid a \in \mathcal{S}^r, b \in \mathbb{R}^{r\times(n-r)} \right\} \tag{2.8}$$

Example 2.3. (The second order cone) The second-order cone in $\mathbb{R}^{n+1}$ is defined as

$$K_{2,n+1} \;=\; \{ (x_0, x) \mid x_0 \geq \| x \|_2 \},$$

and it is self-dual with respect to the usual inner product in $\mathbb{R}^{n+1}$.

Let $F \trianglelefteq K_{2,n+1}$. As $K_{2,n+1}$ is the "lifting" of the unit ball of the norm $\| \;\; \|_2$, all F faces different from $\{ 0 \}$ and $K_{2,n+1}$ must satisfy

$$\begin{aligned} F &= \text{cone}\{ (\| x \|_2, x)^T \} \\ F^{\triangle} &= \text{cone}\{ (\| x \|_2, -x)^T \} \end{aligned} \tag{2.9}$$

for some $x \in \mathbb{R}^n$.

For any two such faces determined by $u, v \in \mathbb{R}^n$ there is a linear map $Q(u, v)$ that sends $(\| u \|_2, u)^T$ to $(\| v \|_2, v)^T$, and $K_{2,n+1}$ to itself. Therefore, we can assume that F is generated by $\bar{x} = (n^{1/2}, e)^T$. Then (2.4) and (2.9) imply

$$\tan(\bar{x}, K_{2,n+1}) = \{ (y_0, y) \mid n^{1/2} y_0 = e^T y \} \tag{2.10}$$

Minimal cones Denote by Feas(P) the feasible set of (P) and assume that it is nonempty. Let

$$\bar{x} \in \text{ri Feas}(P), \quad E := \text{face}(b - A\bar{x}, K).$$

Then for any $y \in \text{Feas}(P)$ there is $z \in \text{Feas}(P)$ with $\bar{x} \in (y, z)$. Hence

$$\text{ri}\, E \ni b - A\bar{x} \in (b - Ay, b - Az) \Rightarrow b - Ay, b - Az \in E,$$

and we obtain that (P) is equivalent to

$$Ax \leq_E b.$$

In other words E is the *maximal* face of K that contains a vector of the form $b - Ax$ in its relative interior. It is called the *minimal cone* of the system (P), corresponding to $b - Ax$, and denoted by mincone$(b - Ax, (P))$.

In general, if S a closed convex cone, (Q) a conic linear system, with

$$A_{i_1} u^{i_1} \leq_S b^{i_1}, \ldots, A_{i_1} u^{i_k} \leq_S b^{i_k}$$

among its constraints, then

$$\text{mincone}(\, (b^{i_1} - A_{i_1} u^{i_1}) + \cdots + (b^{i_k} - A_{i_k} u^{i_k}), (Q))$$

will denote the maximal face of S that contains a vector $(b^{i_1} - A_{i_1} u^{i_1}) + \cdots + (b^{i_k} - A_{i_k} u^{i_k})$ in its relative interior, with all u^{i_j}'s feasible for (Q).

Key assumptions and notation In the rest of the paper, unless otherwise stated, we uphold the assumptions that

$$\boxed{(P) \text{ is feasible; } K \text{ is nice, ie. } F^* = K^* + F^\perp \text{ for all } F \trianglelefteq K}$$

and write

$$\boxed{F_{\min} = \text{mincone}(\, b - Ax, (P)\,)}$$

3 A Facial Reduction Algorithm

Reducing certificates Fix an F face of K such that $F_{\min} \subseteq F \subseteq K$. The feasible solutions of the following conic linear system

$$\left.\begin{aligned} (u, v) &\in K^* \times F^\perp \\ A^*(u+v) &= 0 \\ \langle b, u+v \rangle &= 0 \end{aligned}\right\} \qquad (RED(F))$$

will give a proof when $F_{\min}$ is contained in a smaller face of K.

Theorem 3.1. (Borwein and Wolkowicz) *Assume that (P) is feasible. Then*

(1) For all (u, v) feasible to $(RED(F))$,

$$F_{\min} \subseteq F \cap \{u\}^\perp \subseteq F. \qquad (3.11)$$

(2) If $F_{\min} \subsetneq F$, then there exists (u, v) feasible to $(RED(F))$, such that the second containment in (3.11) is strict.

Proof of (1) Let x be a feasible solution of (P). Then

$$0 = \langle u + v, b - Ax \rangle = \langle u, b - Ax \rangle,$$

proving the first containment; the second is obvious.

Proof of (2) Fix $f \in \operatorname{ri} F$. Then $F_{\min} \neq F$, iff

$$Ax + ft \leq_F b \quad \text{implies} \quad t \leq 0. \tag{3.12}$$

Since the conic system of (3.12) is strictly feasible (with some t sufficiently negative), this implication has a certificate, that is

$$\begin{array}{llllll} \exists\, y \in F^* & \text{st.} & A^* y = 0, & \langle b, y\rangle \leq 0, & \langle f, y\rangle = 1 & \Leftrightarrow \\ \exists\, (u+v) \in K^* + F^{\perp} & \text{st.} & A^*(u+v) = 0, & \langle b, u+v\rangle \leq 0, & \langle f, u+v\rangle = 1. & \end{array}$$

Next, note that $\langle b, y\rangle = 0$ must hold, since $\langle b, y\rangle < 0$ would prove the infeasibility of (P). Finally,

$$\langle f, u+v\rangle = \langle f, u\rangle > 0 \;\Rightarrow\; F \cap \{u\}^{\perp} \subsetneq F.$$

■

If the second containment in (3.11) is strict, we shall say that (u, v) *reduces* the system $Ax \leq_F b$, or it is a *reducing certificate*. Next, we state an algorithm to construct $F_{\min}$; it is a simplified version of the one given by Borwein and Wolkowicz [3].

FACIAL REDUCTION ALGORITHM (A, b, K)

`Input:` A, b, K.
`Output:` $t \geq 0$, $u^0, \dots, u^t \in K^*$ with $F_{\min} = K \cap \{u^0 + \cdots + u^t\}^{\perp}$.
`Invariants:` $F_{\min} \subseteq F_i$,
$F_i = K \cap \{u^0 + \cdots + u^i\}^{\perp}$,
$F_i^{\perp} = \tan(u^0 + \cdots + u^i, K^*)$.

Initialization: Let $(u^0, v^0) = (0, 0)$, $F_0 = K$, $i = 0$.
while $F_{\min} \neq F_i$
Find (u^{i+1}, v^{i+1}) reducing $Ax \leq_{F_i} b$.
Let $F_{i+1} = F_i \cap \{u^{i+1}\}^{\perp}$, $i = i + 1$.
end while
Output $t = i$, $u^0, \dots, u^t$.

Theorem 3.2. *The Facial Reduction Algorithm is finite, and correctly constructs* $F_{\min}$.

Proof It suffices to note the following 3 facts.

- By Lemma 3.1 a (u^{i+1}, v^{i+1}) that reduces F_i can be found exactly if $F_{\min} \neq F_i$.

- All three invariants are trivially satisfied for $i = 0$. Now assume that they are true for $0, \ldots, i$. Then

$$\begin{aligned} F_{\min} \subseteq F_{i+1} &= F_i \cap \{u^{i+1}\}^\perp \\ &= K \cap \{u^0 + \cdots + u^i\}^\perp \cap \{u^{i+1}\}^\perp \\ &= K \cap \{u^0 + \cdots + u^i + u^{i+1}\}^\perp. \end{aligned} \tag{3.13}$$

The last equality follows, as all u^i's are in K^*. Therefore, the first two invariants hold for $i+1$. To prove that the last one does, using (3.13) we obtain

$$\begin{aligned} F_i^\perp &= \left(K \cap \{u^0 + \cdots + u^i\}^\perp\right)^\perp \\ &= \operatorname{face}(u^0 + \cdots + u^i, K^*)^{\triangle\perp} \\ &= \tan(u^0 + \cdots + u^i, K^*), \end{aligned}$$

as required.

- Since F_i is reduced in every step, the number of steps until termination is not more than

$$\begin{aligned} L(A, b, K) := \min \{ &\dim(\mathcal{N}(A^*) \cap \{b\}^\perp), \\ &\text{length of the longest chain of faces in } K \}. \end{aligned} \tag{3.14}$$

■

We shall call the collection of (u^i, v^i)'s found by the algorithm a *facial reduction sequence (FRS)*. By the expression for $F_i^\perp$ above, an FRS will look like

$$\left.\begin{array}{l} (u^0, v^0) = (0, 0) \\ (u^i, v^i) \in K^* \times \tan(u^0 + \cdots + u^{i-1}, K^*) \\ \qquad (i = 1 \ldots, t) \\ u^i + v^i \in \mathcal{N}(A^*) \cap \{b\}^\perp \\ \qquad (i = 1 \ldots, t) \end{array}\right\} \tag{3.15}$$

Example 3.3. With $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, $K = K^* = \mathbb{R}^n_+$, (P) is a linear inequality system. For the instance

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.16}$$

an FRS is

$$u^1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix}^T, u^2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}^T, v^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \end{pmatrix}^T,$$

with the corresponding F_i faces being

$$F_1 = \mathbb{R}^3_+ \times \{0\}^2, F_2 = \mathbb{R}^1_+ \times \{0\}^4.$$

F_2 is the minimal cone of (3.16).

Example 3.4. With $X = \mathbb{R}^m$, $Y = \mathcal{S}^n$, $K = K^* = \mathcal{S}^n_+$, (P) is a semidefinite system. For the instance

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} x_1 + \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} x_2 + \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} x_3 \preceq \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \tag{3.17}$$

an FRS is

$$u^1 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, \; u^2 = \begin{pmatrix}0&0&0\\0&2&0\\0&0&0\end{pmatrix}, \; v^2 = \begin{pmatrix}0&0&-1\\0&0&0\\-1&0&0\end{pmatrix}.$$

The corresponding F_i faces are

$$F_1 = \text{face}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \mathcal{S}^3_+\right), \; F_2 = \text{face}\left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \mathcal{S}^3_+\right).$$

F_2 is the minimal cone of (3.17).

Example 3.5. With $X = \mathbb{R}^m$, $Y = (\mathbb{R}^{n+1})^r$, and $K = K^* = (K_{2,n+1})^r$, (P) is a conic system over a direct product of second order cones. Consider the instance

$$\left[\begin{pmatrix}\sqrt{n}\\e\end{pmatrix} \begin{pmatrix}0\\0\end{pmatrix}\right] x_1 + \left[\begin{pmatrix}0\\a\end{pmatrix} \begin{pmatrix}\sqrt{n}\\e\end{pmatrix}\right] x_2 \leq_K \left[\begin{pmatrix}0\\0\end{pmatrix} \begin{pmatrix}0\\0\end{pmatrix}\right] \tag{3.18}$$

where $a \in \mathbb{R}^n$ is such that $\langle a, e\rangle = 0$, $\| a \|_2^2 = 2n$. Now an FRS is

$$u^1 = \left[\begin{pmatrix}\sqrt{n}\\-e\end{pmatrix} \begin{pmatrix}\sqrt{n}\\-e\end{pmatrix}\right], \; u^2 = \left[\begin{pmatrix}0\\0\end{pmatrix} \begin{pmatrix}\sqrt{n}\\e\end{pmatrix}\right], v^2 = \left[\begin{pmatrix}0\\-a\end{pmatrix} \begin{pmatrix}0\\0\end{pmatrix}\right]$$

The corresponding F_i faces are

$$F_1 = \text{face}\left(\left[\begin{pmatrix}\sqrt{n}\\e\end{pmatrix} \begin{pmatrix}\sqrt{n}\\e\end{pmatrix}\right], K\right), F_2 = \text{face}\left(\left[\begin{pmatrix}\sqrt{n}\\e\end{pmatrix} \begin{pmatrix}0\\0\end{pmatrix}\right], K\right),$$

with F_2 equal to the minimal cone of (3.18).

Remark 3.6. Note that for (3.16) there is an FRS of length 1, namely

$$u^1 = \begin{pmatrix}0&0&0&1&1\end{pmatrix}^T.$$

In fact, this is true for any linear inequality system. If $K = K^* = \mathbb{R}^n$, take an FRS $(\widehat{u}^i, \widehat{v}^i)_{i=0}^t$ satisfying (3.15) and set

$$\widetilde{u}^1 = \widehat{u}^1, \ldots, \widetilde{u}^t = (\widehat{u}^t + \widehat{v}^t) + \alpha_{t-1}\widetilde{u}^{t-1}$$

for sufficiently large $\alpha_1, \ldots, \alpha_{t-1}$. Then

$$\widetilde{u}^1, \ldots, \widetilde{u}^t \in K^*, \text{face}(\widetilde{u}^t, K^*) = \text{face}(\widehat{u}^1 + \cdots + \widehat{u}^t, K^*)$$

Hence the correctness of FRA also leads to the existence of a strictly complementary solution pair for a primal-dual pair of linear programs.

4 The Simplest Extended Dual System

To turn the algorithm for constructing $F_{\min}$ into an extended formulation of $F_{\min}^*$, first notice that the set

$$\{\, (u,v) \,|\, u \in K^*,\ v \in \tan(u, K^*) \,\}$$

is a closed convex cone. For brevity, let $L = L(A, b, K)$. By the previous remark, the structure

$$\left.\begin{array}{l} (u^0, v^0) = 0 \\ (u^i, v^i) \in K^* \times \tan(u^0 + \cdots + u^{i-1}, K^*) \\ \qquad (i = 1 \ldots, L+1) \\ u^i + v^i \in \mathcal{N}(A^*) \cap \{b\}^\perp \\ \qquad (i = 1 \ldots, L) \end{array}\right\} \qquad (EXT)$$

is a conic linear system. Note that the different range for i in the 2 constraints (from 1 to $L+1$ and from 1 to L) is not accidental.

Theorem 4.1. (Representing $F_{\min}^*$) *The following hold.*

$$F_{\min}^{\triangle} = \operatorname{mincone}(\, u^0 + \cdots + u^L, (EXT)\,) \tag{4.19}$$

$$F_{\min}^{\perp} = \{\, v^{L+1} \,|\, (u^i, v^i)_{i=0}^{L+1} \text{ is feasible for } (EXT) \,\} \tag{4.20}$$

$$F_{\min}^{*} = \{\, u^{L+1} + v^{L+1} \,|\, (u^i, v^i)_{i=0}^{L+1} \text{ is feasible for } (EXT) \,\} \tag{4.21}$$

Proof By Theorem 3.1 for all $(u^i, v^i)_{i=0}^{L+1}$ that are feasible for (EXT)

$$F_{\min} \subseteq K \cap \{u^0 + \cdots + u^L\}^\perp = \operatorname{face}(u^0 + \cdots + u^L, K^*)^\triangle,$$

therefore

$$\begin{array}{l} F_{\min}^{\triangle} \supseteq \operatorname{face}(u^0 + \cdots + u^L, K^*)^{\triangle\triangle} = \operatorname{face}(u^0 + \cdots + u^L, K^*), \text{ and} \\ F_{\min}^{\perp} \supseteq \operatorname{face}(u^0 + \cdots + u^L, K^*)^{\triangle\perp} = \tan(u^0 + \cdots + u^L, K^*) \end{array}$$

hold. By Theorem 3.2 equality holds for some feasible $(u^i, v^i)_{i=0}^{R+1}$, thus both (4.19) and (4.20) follow. The statement of (4.21) is implied by (4.20) and $F_{\min}^* = K^* + F_{\min}^\perp$. ■

As an immediate corollary we obtain

Theorem 4.2. (Farkas' lemma without a CQ) *Suppose that (P) is feasible, $c \in X$, $c_0 \in \mathbb{R}$. Then for all x feasible solutions of (P) $\langle c, x\rangle \le c_0$ holds, iff there is $(u^i, v^i)_{i=0}^{L+1}$ such that*

$$(u^i, v^i)_{i=0}^{L+1} \in \operatorname{Feas}(EXT),\ A^*(u^{L+1} + v^{L+1}) = c, \quad \langle b, u^{L+1} + v^{L+1}\rangle \le c_0.$$

■

Naturally, we would prefer to express $F^*_{\min}$ using a conic system in terms of copies of K^*. This is indeed possible for the case of a semidefinite system, ie. when $X = \mathbb{R}^m$, $Y = \mathcal{S}^n$, $K = K^* = \mathcal{S}^n_+$: we must appropriately substite for the "$v \in \tan(u, K^*)$" constraint in (EXT). Consider

$$\left.\begin{array}{rl} (u^0, v^0) = & (0,0) \\ v^i - w^i - (w^i)^T = & 0 \\ \begin{pmatrix} u^0 + \cdots + u^{i-1} & w^i \\ (w^i)^T & \beta_i I \end{pmatrix} \succeq & 0,\ u^i \in psdn, v^i \in \mathcal{S}^n,\ w^i \in \mathbb{R}^{n \times n},\ \beta_i \in \mathbb{R} \\ & (i = 1 \ldots, L+1) \\ u^i + v^i \in & \mathcal{N}(A^*) \cap \{b\}^\perp \\ & (i = 1 \ldots, L\}) \\ \text{with } L = & \min\{\, n(n+1)/2 - m - 1, n\,\} \end{array}\right\} \quad (EXT\text{-}SDP)$$

Corollary 4.3. *Suppose* $X = \mathbb{R}^m$, $Y = \mathcal{S}^n$, $K = K^* = \mathcal{S}^n_+$. *Then*

$$\begin{array}{rl} F^{\triangle}_{\min} = & \operatorname{mincone}(\, u^0 + \cdots + u^L, (EXT\text{-}SDP)\,) \\ F^{\perp}_{\min} = & \{\, v^{L+1} \,|\, (u^i, v^i)_{i=0}^{L+1} \ \textit{is feasible for}\ (EXT\text{-}SDP)\,\} \\ F^{*}_{\min} = & \{\, u^{L+1} + v^{L+1} \,|\, (u^i, v^i)_{i=0}^{L+1} \ \textit{is feasible for}\ (EXT\text{-}SDP)\,\} \end{array}$$

Moreover, let $c \in X$, $c_0 \in \mathbb{R}$. *Then for all* x *feasible solutions of* (P) $\langle c, x\rangle \le c_0$ *holds, iff there is* $(u^i, v^i)_{i=0}^{L+1}$ *such that*

$$(u^i, v^i)_{i=0}^{L+1} \in \operatorname{Feas}(EXT\text{-}SDP),\ A^*(u^{L+1} + v^{L+1}) = c,\ \langle b, u^{L+1} + v^{L+1}\rangle \le c_0.$$

Proof Immediate by noting (cf. (2.8))

$$\tan(u, \mathcal{S}^n_+) = \left\{ w + w^T \;\middle|\; \begin{pmatrix} u & w \\ w^T & \beta I \end{pmatrix} \succeq 0 \quad \text{for some } w \in \mathbb{R}^{n \times n}, \text{and } \beta \in \mathbb{R} \right\}.$$

■

5 Equivalent Extended Dual Systems

So far we have shown that the correctness of (a variant of) FRA immediately leads to the correctness of an EDS. Ramana's original system in [15] is somewhat different from $(EXT\text{-}SDP)$ though. Here we exhibit several equivalent ED systems, one of them being his original. The disaggregated extended dual system is

$$\left.\begin{array}{c} (u^0, v^0) = 0 \\ (u^i, v^i) \in K^* \times \tan(u^{i-1}, K^*) \\ (i = 1 \ldots, L+1) \\ u^i + v^i \in \mathcal{N}(A^*) \cap \{b\}^\perp \\ (i = 1 \ldots, L) \end{array}\right\} \qquad (D\text{-}EXT)$$

Theorem 5.1. *Define*

$$\begin{aligned} G_i &= \text{mincone}(\, u^0 + \cdots + u^i, (EXT)\,), \\ H_i &= \text{mincone}(\, u^i, (D\text{-}EXT)\,), \\ J_i &= \text{mincone}(\, u^i, (D\text{-}EXT)\,). \end{aligned}$$

Then

$$G_i = \; H_i \; = J_i \tag{5.22}$$

$$\begin{aligned} F_{\min}^{\triangle} &= \; \text{mincone}(\, u^0 + \cdots + u^L, (D\text{-}EXT)\,) \\ F_{\min}^{\perp} &= \; \{\, v^{L+1} \,|\, (u^i, v^i)_{i=0}^{L+1} \;\; \textit{is feasible for } (D\text{-}EXT)\,\} \\ F_{\min}^{*} &= \; \{\, u^{L+1} + v^{L+1} \,|\, (u^i, v^i)_{i=0}^{L+1} \;\; \textit{is feasible for } (D\text{-}EXT)\,\} \end{aligned} \tag{5.23}$$

Furthermore, suppose that a set-valued operator

$$\tan' : K^* \rightarrow 2^{\mathbb{R}}$$

is given, with the property

$$\forall v \in \tan(u, K^*) \;\; \alpha_v v \in \tan'(u, K^*) \textit{ for some } \alpha_v \geq 0$$

Then (5.22) holds, if in (EXT) and (D-EXT) tan *is replaced with* $\tan'$.

Proof In (5.22) the inclusions $\supseteq$ are trivial. To see $G_i \subseteq J_i$, let $(\widehat{u}^i, \widehat{v}^i)_{i=0}^{L+1}$ be feasible for (EXT). Then $(\widetilde{u}^i, \widetilde{v}^i)_{i=0}^{L+1}$ defined as

$$(\widetilde{u}^i, \widetilde{v}^i) = (\widehat{u}^0, \widehat{v}^0) + \cdots + (\widehat{u}^i, \widehat{v}^i) \;\; (i = 0 \ldots, L+1)$$

is feasible for $(D\text{-}EXT)$. Thus (5.23) follows as well.

Take $(\widetilde{u}^i, \widetilde{v}^i)_{i=0}^{L+1}$ feasible for (EXT). Then there is $\alpha_i \, (i = 1, \ldots, L+1)$ such that $(\alpha_i \widetilde{u}^i, \alpha_i \widetilde{v}^i)_{i=0}^{L+1}$ is feasible, if in (EXT) we replace "tan" with "tan′". Hence this replacement does not affect mincone$(\, u^0 + \cdots + u^L, (D\text{-}EXT)\,)$. The proof is the same for $(D\text{-}EXT)$. ■

For the semidefinite case, we can thus recover Ramana's original system

$$\left.\begin{array}{rl} (u^0, v^0) = & (0,0) \\ v^i - w^i - (w^i)^T = & 0 \\ \begin{pmatrix} u^{i-1} & w^i \\ (w^i)^T & I \end{pmatrix} \succeq & 0, \; u^i \in \mathcal{S}^n_+, v^i \in \mathcal{S}^n, \; w^i \in \mathbb{R}^{n \times n} \\ & (i = 1 \ldots, L+1) \\ u^i + v^i \in & \mathcal{N}(A^*) \cap \{b\}^\perp \\ & (i = 1 \ldots, L) \\ \text{with } L = & \min\{\, n(n+1)/2 - m - 1, n \,\} \end{array}\right\} \qquad (RAM)$$

and the essence of his result as

Corollary 5.2. *Suppose* $X = \mathbb{R}^m$, $Y = \mathcal{S}^n$, $K = K^* = \mathcal{S}^n_+$. *Then*

$$\begin{array}{rl} F_{\min}^{\triangle} = & \operatorname{mincone}(\, u^0 + \cdots + u^L, \; (RAM)\,) \\ F_{\min}^{\perp} = & \{\, v^{L+1} \,|\, (u^i, v^i)_{i=0}^{L+1} \;\textit{is feasible for}\, (RAM) \,\} \\ F_{\min}^{*} = & \{\, u^{L+1} + v^{L+1} \,|\, (u^i, v^i)_{i=0}^{L+1} \;\textit{is feasible for}\, (RAM) \,\} \end{array}$$

Moreover, let $c \in X$, $c_0 \in \mathbb{R}$. *Then for all* x *feasible solutions of* (P) $\langle c, x \rangle \leq c_0$ *holds, iff there is* $(u^i, v^i)_{i=0}^{L+1}$ *such that*

$$(u^i, v^i)_{i=0}^{L+1} \;\textit{is feasible for}\, (RAM), \quad A^*(u^{L+1} + v^{L+1}) = c, \quad \langle b, u^{L+1} + v^{L+1} \rangle \leq c_0.$$

Proof The system (RAM) is just $(D\text{-}EXT)$, with

$$\tan'(u, \mathcal{S}^n_+) = \left\{ w + w^T \;\middle|\; \begin{pmatrix} u & w \\ w^T & I \end{pmatrix} \succeq 0 \quad \text{for some } w \in \mathbb{R}^{n \times n} \right\}.$$

■

6 Representation of the minimal cone, and its relatives

We have seen that $F_{\min}$, the minimal cone of the conic linear system

$$Ax \leq_K b \qquad (P)$$

has the following useful properties:

(i) The sets $F_{\min}^{\perp}$ and $F_{\min}^{*}$ have an extended formulation; that is, they are the projection of some conic linear systems in a higher dimensional space. The only nontrivial conic constraints in these systems are of the form $u \geq_{K^*} 0$, and $v \in \tan(u, K^*)$.

(ii) When $K = K^*$ is the semidefinite cone, there is a representation in terms of only $K = K^*$ itself.

Several related questions arise naturally.

(1) For what other related sets is there a representation as described in (i) and (ii) ? In particular, is there one for $F_{\min}$ itself, and $F_{\min}^{\triangle}$?

(2) How "long" does such a representation have to be ? For $F_{\min}^{\perp}$, and $F_{\min}^{*}$, is there one specified with less data, than in (4.20) and (4.21) ?

This section will provide partial answers and some conjectures. First, note that when a fixed $\bar{s} \in \operatorname{ri} F_{\min}$ is given, then obviously

$$F_{\min} = \{ s \mid 0 \leq_K s \leq_K \alpha\bar{s} \text{ for some } \alpha \geq 0 \}.$$

It is not hard to see that $F_{\min}$ can be represented without the explicit knowledge of $\bar{s}$, since

$$F_{\min} = \{ s \mid 0 \leq_K s \leq_K \alpha b - Ax \text{ for some } x, \text{ and } \alpha \geq 0 \} \tag{6.24}$$

This result was obtained by Freund [5], based on the article by himself, Roundy and Todd [16]; of course, (6.24) holds without any assumption on K.

If K is nice, then by (4.19) we obtain

$$F_{\min}^{\triangle} = \{ u \mid 0 \leq_{K^*} u \leq_{K^*} (u^0 + \ldots u^L) \\ \text{for some } (u^i, v^i)_{i=0}^{L+1} \text{ feasible for } (EXT) \}$$

Again, the substitution for the "*tan*" constraint can be used when $K = K^* = \mathcal{S}_+^n$. This gives a partial answer to (1) above; it still remains to be seen, whether there are more compact extended formulations for $F_{\min}^{\triangle}, F_{\min}^{\perp}, F_{\min}^{*}$.

As to (2), it would be interesting to see, for what other nice cones is there an extended formulation for the set

$$\{ (u, v) \mid u \in K^*, \, v \in \tan(u, K^*) \}$$

in terms of K^* (or for the variant with "tan′" replacing "tan"). By Theorem 4.1 conic linear systems over such cones would also have Ramana-type (ie. expressed only in terms of K^*) extended dual systems.

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