On the Closedness of the Linear Image of a Closed Convex Cone

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The Problem

We are given a closed, convex cone, and a linear mapping. Under what conditions is the image of the cone closed?

- A very simple question in convex analysis $\rightarrow$ interesting on its own right.
- Fundamental in studying duality theory.

The setup

Let

- K be a closed, convex cone, ($x \in K$, $\lambda \geq 0 \Rightarrow \lambda x \in K$).
- $K^* = \{y \mid \langle y, s \rangle \geq 0 \; \forall s \in K\}$ the *dual* of K.
- M a linear map, M^* its adjoint (transpose).

The question

- Under what conditions is M^*K^* is closed ?

Classical results

- **If** K is polyhedral,
- **Or** $\mathcal{R}(M) \cap \mathrm{ri}\ K \neq \emptyset$ ("Slater-condition"),
- **Then** M^*K^* is closed.

More recent results

- Waksman and Epelman (76): a simple condition, that reduces to the classical ones in most important cases.
- Auslender (96): a more complicated, necessary and sufficient condition for arbitrary closed convex sets.
- Bauschke and Borwein (99): a necessary and sufficient condition for the continuous image of a closed convex cone, in terms of the CHIP property.
- Ramana (98): An extended dual for semidefinite programs, without any CQ: related to work of Borwein and Wolkowicz in 84 on facial reduction.

Outline of main results

We provide simple, equivalent conditions that are

- *necessary* for all cones,
- *necessary and sufficient* for a large class of cones, that we call **nice** cones. (Technical condition, more about it later).
- **Fact:** Most cones occurring in optimization (polyhedral, semidefinite, quadratic, lp-norm cones etc.) are nice.

Some important basics

- C convex set. $\text{dir}\,(x, C) := \{\, y \,|\, x + \alpha y \in C \text{ for some } \alpha > 0 \,\}$: the *feasible directions* at x in C.

- Fact: $\text{dir}\,(x, C)$ is a convex cone, but it may not be closed!

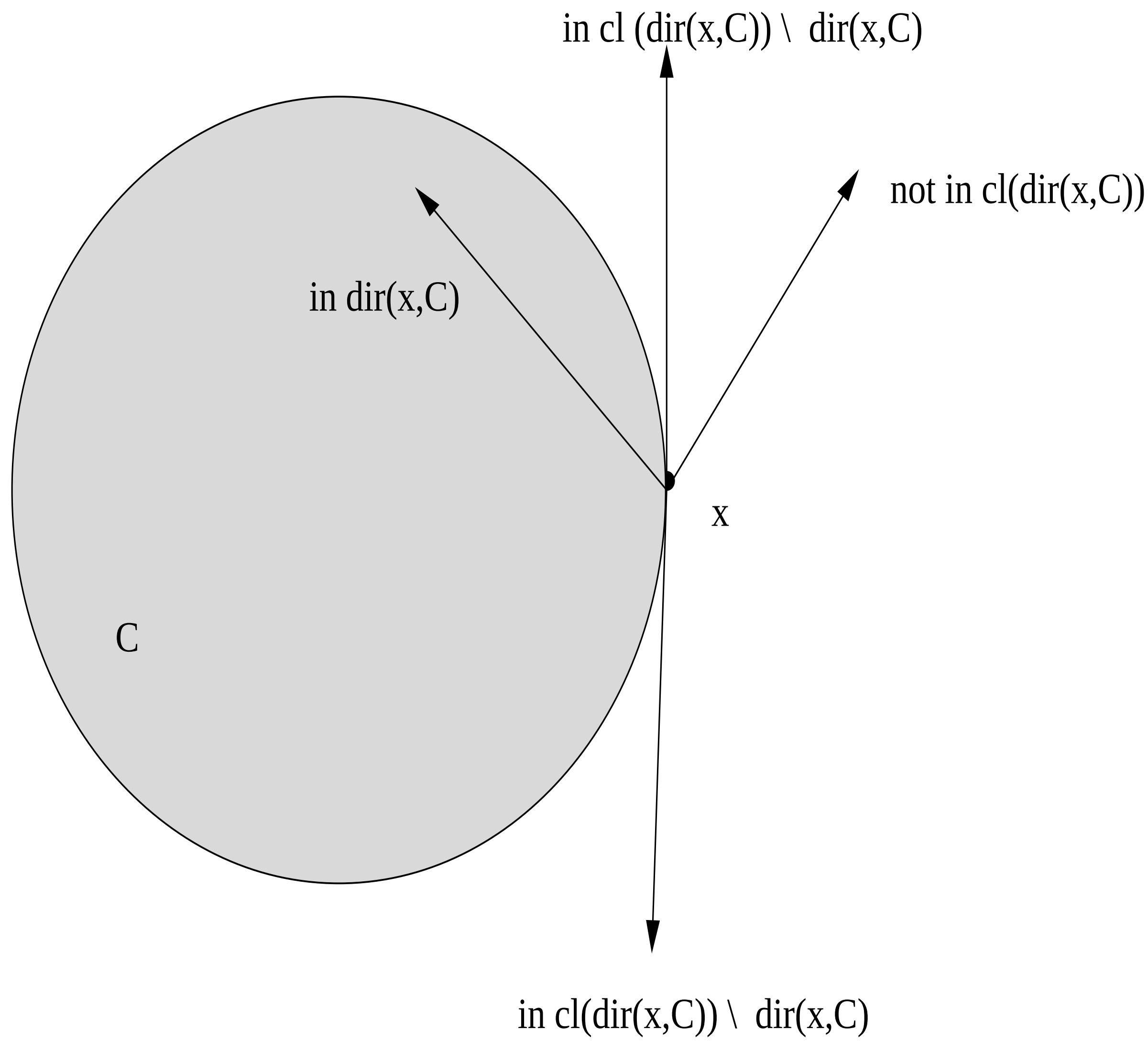

Figure 1: Feasible directions

Main Result, Part 1

Let K be a closed cone, M a linear map, $x \in \mathrm{ri}\,(\mathcal{R}(M) \cap K)$ (nonneg. orthant: max # of nonzeros; semidef. cone: max. rank).

Then

$$M^* K^* \text{ is closed} \quad \Rightarrow$$
$$\mathcal{R}(M) \cap \mathrm{cl}\ \mathrm{dir}\ (x, K) \quad = \quad \mathcal{R}(M) \cap \mathrm{dir}\ (x, K)\ (\textbf{Condition 1})$$

If K is nice, then $\Leftrightarrow$ is true.

Obviously,

K is polyhedral **or** $x \in \mathrm{ri}\ K \Rightarrow \mathrm{dir}\ (x, K)$ is closed $\Rightarrow$ Condition 1.

Example 1 $K = K^* = \mathcal{S}^2_+ = 2 \times 2$ psd matrices.

$$
\begin{aligned}
M \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= z_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
x &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \operatorname{ri}(\mathcal{R}(M) \cap K) \\
y &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{R}(M) \cap (\text{cl dir } (x, K) \setminus \text{dir } (x, K))
\end{aligned}
$$

The x and y are **certificates** of the nonclosedness of $M^* K^*$.

Indeed, we can check the nonclosedness of M^*K^* directly:

- $M^* \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} a \\ 2c \end{bmatrix}.$
- Then $\begin{bmatrix} 0 \\ 2 \end{bmatrix} \in \mathrm{cl}\,(M^* \mathcal{S}^2_+)$, since $M^* \begin{bmatrix} \epsilon & 1 \\ 1 & 1/\epsilon \end{bmatrix} = \begin{bmatrix} \epsilon \\ 2 \end{bmatrix}.$
- But $\begin{bmatrix} 0 \\ 2 \end{bmatrix} \notin M^* \mathcal{S}^2_+$, since $\begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix} \notin \mathcal{S}^2_+$ for any b.

Next: some equivalent variants of Condition 1. Let

$$\begin{aligned} x &\in \operatorname{ri}(\mathcal{R}(M) \cap K) \\ F &= \text{the minimal face of } K \text{ that contains } x \\ F^{\perp} &= \{\, y \mid y^T x = 0 \ \ \forall x \in F \,\} \quad \text{(a subspace)} \\ F^{\triangle} &= K^* \cap F^{\perp} \quad \text{(a face of } K^*\text{)} \end{aligned}$$

$F^{\triangle}$ is called the complementary (conjugate) face of F.

Example If $K = K^* = \mathcal{S}^n_+$, a typical F, $F^\perp$, and $F^\triangle$ look like

$$\begin{aligned}
F &= \left\{ \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \mid U \succeq 0 \right\} \\
F^\perp &= \left\{ \begin{bmatrix} 0 & V \\ V^T & W \end{bmatrix} \mid V, W \text{ free} \right\} \\
F^\triangle &= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix} \mid W \succeq 0 \right\}
\end{aligned}$$

Main Result, Part 2

Let K, M and F be as before. Then

- M^*K^* is closed $\Rightarrow M^*F^{\triangle} = M^*F^{\perp}$ **(Condition 2)**

If K is nice, then $\Leftrightarrow$ is true.

Condition 2 rephrased for $K =$ the nonnegative orthant.
Suppose that in

$$\begin{aligned} M_0 y &\geq 0 \\ M_+ y &\geq 0 \end{aligned}$$

the first group of inequalities always hold at equality, and it is maximal w.r.t. this property (ie. $\exists \bar{y} \ : \ M_0 \bar{y} = 0, \, M_+ \bar{y} > 0$).

Then Condition 2 $\Leftrightarrow$

$$\{ \, y^T M_0 \, | \, y \geq 0 \, \} \quad = \quad \{ \, y^T M_0 \, | \, y \text{ free} \, \}$$

Main Result, Part 3

$F^{\triangle} := K \cap F^{\perp}$: the *complementary* face of F. Then

$M^* K^*$ is closed $\Rightarrow$

(1) $\exists\, u \in \operatorname{ri} F^{\triangle} \cap \mathcal{N}(M^*)$, **and**

(2) $M^*(\tan(u, K^*)) = M^*(\operatorname{lin}\ F^{\triangle})$.

If K is nice, then $\Leftrightarrow$ is true.

(1) $\Leftrightarrow$ x and u are a **strictly complementary pair**, that is, $x \in \mathcal{R}(M) \cap \operatorname{ri} F$ and $u \in \mathcal{N}(M^*) \cap \operatorname{ri} F^{\triangle}$.

- (1) for K polyhedral : true by Goldman-Tucker.
- (2) for K polyhedral : the tangent space and the linear span are the same.

Example 3

$$Mz = z_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Certificates of closedness:

- (SC) points in $\mathcal{R}(M) \cap K$ and $\mathcal{N}(M^*) \cap K^*$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $\operatorname{lin}\ F^{\triangle}$ and $\tan(u, K^*)$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} \qquad \begin{bmatrix} 0 & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$$

Hence $M^* K^*$ is closed.

Computational relevance of Condition 3

To verify that M^*K^* is closed, we need to check:

- The pair (x, u) is strictly complementary, and
- Two **subspaces** are equal. This is easy, as opposed to checking the equality of two arbitrary sets.

Hence, if $K = K^* = \mathcal{S}^n_+$, we can **verify** the closedness of M^*K^* in polynomial time, in the real number model of computing.

The examples so far were easy ... But:

Example 4 Using Condition 3, it is easy to verify the closedness of $M^* \mathcal{S}^4_+$, where

$$M^* : \mathcal{S}^4_+ \ni Y \longrightarrow \begin{bmatrix} y_{11} \\ 2y_{12} - y_{22} + y_{33} + 2y_{24} \\ 2y_{13} + y_{22} - y_{33} \\ 2y_{14} + 2y_{23} \end{bmatrix}$$

The verification seems quite hard **without** Condition 3.

So, what are nice cones?

Definition K is *nice*, if for all faces F of K, $F^* = K^* + F^\perp$.

For $K = K^* =$ nonnegative orthant:

$$
\begin{array}{rcl}
F & = & \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \geq 0 \right\} \\
F^* & = & \left\{ \begin{bmatrix} z \\ y \end{bmatrix} \mid z \geq 0, \, y \, \text{free} \right\} \\
F^\perp & = & \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \, \text{free} \right\}
\end{array}
$$

They first appear in a paper by Borwein and Wolkowicz in 1980.
Niceness seems like a reasonable "relaxation" of polyhedrality.

Theorem

1. K is nice $\Rightarrow$ K is facially exposed.
2. K is facially exposed, and for all faces F of K, F^* is facially exposed $\Rightarrow$ K is nice.

Figure 2: A facially **not** exposed convex set

Conclusion, and further work

- Very simple, necessary condition for the closedness of the image of a closed convex cone;
- Exact for most relevant cones occurring in optimization.
- Certificates for
  - Nonclosedness of the image.
  - Closedness of the image.
- Ongoing work:
  - What are nice cones?
  - What about cones, which are not nice ?
  - Applications . . .